1.0 Planck Distribution

Plank's distribution law for the energy flux of radiation emitted from a black body at temperature T per unit area per unit time, as a function of frequency, is:-

$$P(T,\nu) = \frac{2\pi h}{c^2}\frac{\upsilon^3}{exp\left(\frac{h\upsilon}{kT}\right)-1} \quad (1.1)$$

If we integrate (contour integration) this over all frequencies we get the total radiated energy flux U (units W m^{-2}) i.e.

$$U = \frac{2\pi h}{c^2}\int_0^\infty \frac{\upsilon^3}{exp\left(\frac{h\upsilon}{kT}\right)-1}d\upsilon = \frac{2}{15}\frac{\pi^5 k^4}{h^3c^2}T^4 \equiv \sigma T^4 \quad (1.2)$$

The last step in (1.2) produces the Stefan-Boltzmann law where σ is the Stefan-Boltzmann constant (= 5.67x10^{-8} W m^{-2} K^{-4}). Note that the total energy density

$$E = \frac{8\pi h}{c^3}\int_0^\infty \frac{\upsilon^3}{exp\left(\frac{h\upsilon}{kT}\right)-1}d\upsilon = \frac{8}{15}\frac{\pi^5 k^4}{(hc)^3}T^4 = \frac{4}{c}\sigma T^4 \quad (1.2a)$$

Now, given that frequency and wavelength are related by $\nu = \frac{c}{\lambda}$ which implies:-

$$d\nu = -\frac{c}{\lambda^2}d\lambda \quad (1.3)$$

we can write:-

$$U = \frac{2\pi h}{c}\int_0^\infty \frac{c^3}{\lambda^5\left(exp\left(\frac{hc}{\lambda kT}\right)-1\right)}d\lambda = \frac{2}{15}\frac{\pi^5 k^4}{h^3c^2}T^4 \quad (1.4)$$

$$E = \frac{8\pi h}{c^2}\int_0^\infty \frac{c^3}{\lambda^5\left(exp\left(\frac{hc}{\lambda kT}\right)-1\right)}d\lambda = \frac{8}{15}\frac{\pi^5 k^4}{(hc)^3}T^4 = \frac{4}{c}\sigma T^4 \quad (1.4a)$$

So, Planck's distribution law for the energy flux of radiation emitted from a black body at temperature T per unit area per unit time, as a function of wavelength, is:-

$$P(T,\lambda) = \frac{2\pi hc}{\lambda^5}\frac{1}{\left(exp\left(\frac{hc}{\lambda kT}\right)-1\right)} \quad (1.5)$$

Note that the functions $P(T,\nu)$ and $P(T,\lambda)$ are different and when plotting the distribution against wavelength rather than frequency we are giving the x axis a non-linear stretch (defined by equation (1.3)), therefore the peak frequency in $P(\nu)$ and peak wavelength in $P(T,\lambda)$ are, somewhat counter intuitively, not related by $\nu = \frac{c}{\lambda}$.

In prism/grating spectroscopy we are measuring $P(T,\lambda)$ (presumably a photoelectric effect experiment could directly measure $P(T,\nu)$) so we must use equation (1.5) to generate a black body curve.

We can look for the peak positions in the two distribution functions (ν_p and λ_p) by differentiating equations (1.1) and (1.5) then setting both differentials to zero. For $P(T,\nu)$ we obtain:-

$$\frac{\nu_p exp\left(\frac{h\upsilon_p}{kT}\right)\frac{h}{kT}}{exp\left(\frac{h\upsilon_p}{kT}\right)-1} = 3 \qquad (1.6)$$

whilst for $P(T,\lambda)$ we obtain:-

$$\frac{\frac{c}{\lambda_p} exp\left(\frac{hc}{\lambda_p kT}\right)\frac{h}{kT}}{exp\left(\frac{hc}{\lambda_p kT}\right)-1} = 5 \qquad (1.7)$$

Equation (1.7) can be written as:-

$$\frac{\nu_p exp\left(\frac{h\upsilon_p}{kT}\right)\frac{h}{kT}}{exp\left(\frac{h\upsilon_p}{kT}\right)-1} = 5 \qquad (1.7a)$$

with $\nu_p \equiv \frac{c}{\lambda_p}$. Now equations (1.6) and (1.7a) are identical except for the number on the right hand side and so both cannot be true simultaneously. In an attempt to "square the circle" we could compromise and solve:-

$$\frac{\nu_p exp\left(\frac{h\upsilon_p}{kT}\right)\frac{h}{kT}}{exp\left(\frac{h\upsilon_p}{kT}\right)-1} = 4 \qquad (1.8)$$

but this is unphysical. It's better just to accept that $\nu_p \neq \frac{c}{\lambda_p}$ in which case we need to fit the distribution given by equation (1.5) to our prism/grating derived data and calculate the peak from equation (1.7), note as this is a transcendental equation it needs to be iterated. Dropping the p subscript and letting $a \equiv \frac{hc}{kT}$ then we can define the following functions from equation (1.7):-

$$F(\lambda) \equiv \frac{a exp\left(\frac{a}{\lambda}\right)}{\lambda\left[exp\left(\frac{a}{\lambda}\right)-1\right]} - 5 \qquad (1.9)$$

$$\frac{dF(\lambda)}{d\lambda} \equiv F'(\lambda) = \frac{F(\lambda)+5}{\lambda}\left\{F(\lambda) + 4 - \frac{a}{\lambda}\right\} \qquad (1.10)$$

then, to iterate guess a λ_0 and repeatedly calculate:-

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)} \qquad (1.11)$$

2.0 Thermal Spectral Line Broadening

The probability of a fluctuation ΔE from the mean energy in a system in thermal equilibrium at absolute temperature T is:-

$$P(\Delta E) = P_0 e^{-\frac{\Delta E}{kT}} \qquad (2.1)$$

As $\Delta E = \frac{1}{2} m\Delta V^2$ this corresponds to an atomic velocity fluctuation (ΔV) probability of:-

$$P(\Delta V) = P_0 e^{-\frac{m\Delta v^2}{2kT}} \qquad (2.2)$$

Which, as $\Delta V = c\frac{\Delta\lambda}{\lambda_0}$, in turn corresponds to a Doppler shift ($\Delta\lambda$) probability of:-

$$P(\Delta\lambda) = P_0 e^{-\frac{mc^2\Delta\lambda^2}{2kT\lambda_0^2}} \tag{2.3}$$

i.e. a Gaussian distribution:-

$$P(\lambda, \lambda_0) = P_0 e^{-\frac{(\lambda-\lambda_0)^2}{2\sigma^2}} \tag{2.4}$$

with $\sigma = \sqrt{\frac{kT}{mc^2}}\lambda_0$ and $P_0 = \frac{1}{\sigma\sqrt{2\pi}}$

FWHM: $P(\Delta\lambda_{\mathrm{FWHM}}) = \frac{P_0}{2}$ i.e.:-

$$\frac{1}{2} = e^{-\frac{\Delta\lambda_{\mathrm{FWHM}}^2}{2\sigma^2}} \tag{2.5}$$

Therefore

$$\mathrm{FWHM} = 2\Delta\lambda_{\mathrm{FWHM}} = \sqrt{8\,ln2}\,\sigma \tag{2.6}$$

3.0 Pressure Spectral Line Broadening

Spectral line widths are affected by pressure, the more frequent atomic collisions are the more a given spectral line will be broadened. This is a resonance process and follows a Lorentzian distribution:-

$$L(\nu) = \frac{1}{\pi}\left\{\frac{(\Gamma/4\pi)}{(\upsilon-\upsilon_0)^2+(\Gamma/4\pi)^2}\right\} \tag{3.1}$$

where $\int_{-\infty}^{+\infty} L(\nu)d\nu = 1$ and $\Gamma = \gamma + \Lambda\nu_{col}, \gamma$ is a quantum mechanical factor (which I will assume is negligible), ν_{col} is the average collision frequency and Λ is a dimensionless constant of proportionality.

We will assume that:-

$$\Lambda = \frac{4\pi}{\alpha^2} \tag{3.2}$$

The reason for this choice is depicted in figure 1 where we have illustrated the mechanism behind energy exchange, mediated by a photon, between colliding atoms. This is a two vertex Feynman diagram thus incurring a dimensionless factor of $\frac{4\pi}{\alpha^2}$ in its probability.

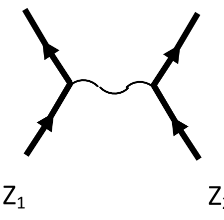

Figure 1: Feynman diagram for atomic interaction

The assignment (3.2) is at this point un-substantiated but its validity will be tested when we come to compare predictions of Solar properties with known values.

Expressing (3.1) in terms of wavelength we obtain:-

$$L(\lambda) \approx k\left(\frac{\lambda_0^2}{\pi c}\right)\left\{\frac{\left(\lambda_0^2\Gamma/_{4\pi c}\right)}{(\lambda-\lambda_0)^2+\left(\lambda_0^2\Gamma/_{4\pi c}\right)^2}\right\} \qquad (3.3)$$

where k is to be determined to ensure $\int_{-\infty}^{+\infty} L(\lambda)d\lambda = 1$. The approximation sign arises from the fact that whilst $L(\nu)$ is symmetric about υ_0 a strict conversion to wavelength would be un-symmetric. In the, symmetric, expression for $L(\lambda)$ the terms like $\lambda\lambda_0$ have been set equal to λ_0^2 Errors introduced should be small if, as they are, line widths are small compared to λ_0.

Making the substitution:-

$$(\lambda - \lambda_0) = \left(\lambda_0^2\Gamma/_{4\pi c}\right)\tan\theta$$

We can deduce that:-

$$L(\lambda) = \frac{1}{\pi}\left\{\frac{\left(\lambda_0^2\Gamma/_{4\pi c}\right)}{(\lambda-\lambda_0)^2+\left(\lambda_0^2\Gamma/_{4\pi c}\right)^2}\right\} \qquad (3.4)$$

We can express ν_{col} in terms of a mean-free-path (l) and mean velocity ($\bar{\upsilon}$) via:-

$$\nu_{col} = \bar{\upsilon}/_l \qquad (3.5)$$

therefore:-

$$\Gamma \approx \Lambda\nu_{col} = \frac{4\pi\bar{\upsilon}}{\alpha^2 l} \qquad (3.6)$$

Note: the Kinetic Theory of Gasses tells us that $\bar{\upsilon} = \left(\frac{3kT}{m}\right)^{1/2}$ so if we know the temperature of a star's photosphere then the fitting parameter Γ gives us a value for the mean-free-path l.

It can be, relatively easily, deduced that:-

$$FWHM = \frac{\Gamma}{2\pi}\frac{\lambda_0^2}{c} = \frac{2}{\alpha^2 l}\left(\frac{3kT}{m}\right)^{1/2}\frac{\lambda_0^2}{c} \qquad (3.7)$$

4.0 Rotational Spectral Line Broadening (Uniformly Emitting Oblate Spheroid)

Using oblate spheroidal co-ordinates, let a point on the surface of the star have position vector (relative to its centre):-

$$\underline{r} = a\left(\cosh\xi\cos\eta\cos\varphi\ \underline{i} + \cosh\xi\cos\eta\sin\varphi\ \underline{j} + \sinh\xi\sin\eta\ \underline{k}\right) \qquad (4.1)$$

$a > 0,\ \ \xi \geq 0,\ \ -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2},\ \ 0 \leq \varphi \leq 2\pi$ and the volume integral can be written as:-

$$V = \iiint h_\xi h_\eta h_\varphi d\xi d\eta d\varphi$$

With $h_\xi = h_\eta = a(\sinh^2\xi + \sin^2\eta)^{1/2}$ and $h_\varphi = a\cosh\xi\cos\eta$

In this analysis I will assume that the star is rotating about the z axis i.e. $\underline{\omega} = \omega\underline{k}$ and we are observing the star from within the $\underline{i}, \underline{k}$ plane at a angle ϑ relative to the x axis i.e from the direction:-

$$\underline{\hat{d}} = \cos\vartheta\ \underline{i} + \sin\vartheta\,\underline{k} \quad \text{where } 0 \le \vartheta \le \frac{\pi}{2} \tag{4.2}$$

The degree of oblateness is determined by the coordinate ξ together with the constant a. Spherical symmetry results if $\xi \gg 1$ and $a \ll 1$ such that $r_{star} = a\cosh\xi \cong a\sinh\xi$. Disk symmetry results if $\xi = 0$ in which case $r_{disk} = a$. At intermediate values of ξ the equatorial radius (r_E) and the polar radius (r_P) are related by:-

$$\frac{r_E}{r_P} = \coth\xi \tag{4.3}$$

We will be wanting to set a maximum stellar equatorial surface velocity as a fraction of c, this can be achieved by setting $0 \le \frac{\omega a\cosh\xi}{c} < 1$. Together with the desired ratio of radii this determines values for ω and ξ at the surface of the star assuming $a \equiv 1$ stellar radius unit.

The first task is to determine the unit normal $(\underline{\hat{n}})$ to a elemental spheroidal surface area at position $\underline{r}$. As the co-ordinate system is orthogonal curvilinear the normal can be calculated from:-

$$\underline{\hat{n}} = \frac{1}{h_\xi}\frac{d\underline{r}}{d\xi} = \frac{1}{(\sinh^2\xi + \sin^2\eta)^{\frac{1}{2}}}\left\{\sinh\xi\cos\eta\cos\varphi\underline{i} + \sinh\xi\cos\eta\sin\varphi\underline{j} + \cosh\xi\sin\eta\underline{k}\right\}$$

Therefore, assuming uniform intensity emitted per unit area (I_0), we can determine the total received intensity from:-

$$I = I_0 \iint \underline{\hat{d}}\,.\,\underline{\hat{n}}\, h_\eta h_\varphi d\eta d\varphi = \iint I(\eta,\varphi)\, d\eta d\varphi \tag{4.4}$$

Where

$$I(\eta,\varphi) = I_0\cosh\xi\cos\eta\{\sinh\xi\cos\eta\cos\varphi\cos\vartheta + \cosh\xi\sin\eta\sin\vartheta\}$$

We want to integrate $I(\eta,\varphi)$ along a contour of constant line of sight velocity, therefore we need to calculate the velocity at any point from:-

$$\underline{v} = \underline{\omega} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ \cosh\xi\cos\eta\cos\varphi & \cosh\xi\cos\eta\sin\varphi & \sinh\xi\sin\eta \end{vmatrix} \tag{4.5}$$

From which we obtain:-

$$\underline{v} = -\omega\cosh\xi\cos\eta\sin\varphi\underline{i} + \omega\cosh\xi\cos\varphi\underline{j} \tag{4.6}$$

Therefore the line of sight velocity is:-

$$v = \underline{v}.\underline{\hat{d}} = -\omega\cosh\xi\cos\eta\sin\varphi\cos\vartheta = \frac{\Delta\lambda c}{\lambda_0} \quad \text{(Doppler shift)} \tag{4.7}$$

For a given λ and λ_0 define the constant K as:-

$$K \equiv \cos\eta \sin\varphi = \frac{-(\lambda-\lambda_0)}{\lambda_0\left(\frac{\omega\cosh\xi}{c}\right)\cos\vartheta} \tag{4.8}$$

(which implies $K \leq 1$ and that linewidth is proportional to wavelength) and therefore:-

$$\sin\varphi = \frac{K}{\cos\eta} \text{ and } \cos\varphi = \pm\left[1-\left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}} \tag{4.9}$$

These last equations define the contour along which we wish to evaluate $I(\eta,\varphi)$i.e. it relates η and φ so we can determine $I\big(\eta,\varphi(\eta)\big) \equiv I(\eta)$ to be:-

$$I(\eta) = I_0 \cosh\xi \cos\eta \left\{\sinh\xi \cos\eta \left[1-\left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}} \cos\vartheta + |\mathrm{K}| \cosh\xi \tan\eta \sin\vartheta\right\} \tag{4.10}$$

For a given value of K (i.e. given $\Delta\lambda$) we can evaluate the following integral to obtain the received intensity at a particular $\Delta\lambda$:-

$$I(K) = \int_{\eta_{min}}^{\eta_{max}} I(\eta)\frac{dl}{d\eta}d\eta \tag{4.11}$$

where dl is the line element given by:-

$$\frac{dl}{d\eta} = \left[\left(h_\eta\right)^2 + \left(h_\varphi \frac{d\varphi}{d\eta}\right)^2\right]^{\frac{1}{2}} \tag{4.12}$$

along the contour we have:-

$$\frac{d\varphi}{d\eta} = -\frac{\sin\eta\sin\varphi}{\cos\eta\cos\varphi} \tag{4.13}$$

therefore:-

$$\frac{dl}{d\eta} = \left[\sinh^2\xi + \sin^2\eta + \cosh^2\xi\cos^2\eta\left(\frac{d\varphi}{d\eta}\right)^2\right]^{\frac{1}{2}} \tag{4.14}$$

Note: this reduces to the spherical limit as both $\sinh^2\xi$ and $\cosh^2\xi$ become very much greater than $\cos^2\eta$. Substituting we obtain:-

$$\frac{dl}{d\eta} = \left[\sinh^2\xi + \sin^2\eta + \frac{\cosh^2\xi\sin^2\eta\left(\frac{K}{\cos\eta}\right)^2}{1-\left(\frac{K}{\cos\eta}\right)^2}\right]^{\frac{1}{2}} \tag{4.15}$$

Therefore we can write:-

$$I(K) = I_0\cosh\xi \int_{\eta_{min}}^{\eta_{max}} \left\{\sinh\xi\cos^2\eta\left[1-\left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}}\cos\vartheta + \cosh\xi|K|\sin\eta\sin\vartheta\right\}\left[\sinh^2\xi + \sin^2\eta + \frac{\cosh^2\xi\sin^2\eta\left(\frac{K}{\cos\eta}\right)^2}{1-\left(\frac{K}{\cos\eta}\right)^2}\right]^{\frac{1}{2}} d\eta \tag{4.16}$$

We now need to determine the limits of integration, the limits are reached when the contour hits the visible limb of the star. On the limb we have the condition:-

$$\underline{\hat{n}}.\underline{\hat{d}} = (\cos\vartheta\ \underline{i} + \sin\vartheta\ \underline{k}).\left(\sinh\xi\cos\eta\cos\varphi\ \underline{i} + \sinh\xi\cos\eta\sin\varphi\ \underline{j} + \cosh\xi\sin\eta\ \underline{k}\right) = 0$$

which implies:-

$$\sinh\xi\cos\eta\cos\varphi\cos\vartheta + \cosh\xi\sin\eta\sin\vartheta = 0 \qquad (4.17)$$

substituting for φ and rearranging we have:-

$$\cos\eta = \pm\sqrt{\frac{(\coth\xi\tan\vartheta)^2+K^2}{1+(\coth\xi\tan\vartheta)^2}} \qquad (4.18)$$

This needs to be solved for the two limits, define $-\frac{\pi}{2} \le \eta_1 \le 0$ and $\frac{\pi}{2} \le \eta_2 \le \frac{3\pi}{2}$. One limb intersection (η_2) will be in the northern hemisphere and the other in the southern. If, as will generally be the case $\eta_2 > \frac{\pi}{2}$ (exception $\vartheta = \frac{\pi}{2}$), we will need to split the integral such that $I = I_1 - I_2$ where I_1 is integrated between limits η_1 and η_{max} whilst I_2 is integrated between limits η_{max} and η_2 where η_{max} is calculated from $K = \cos\eta\sin\varphi$ with $\varphi = \frac{\pi}{2}$ i.e.:-

`

$$\eta_{max} = \cos^{-1}|K| \qquad (4.19)$$

4.1 Simulating a uniform disk

We can, for a very oblate spheroid, calculate the emission from an annular disk by limiting the value of η_{max} to a value smaller than $\cos^{-1}|K|$. If we imagine a "core" (around the $\underline{k}$ axis) taken out of the oblate spheroid

$$(r_{core})^2 = (\cosh\xi\cos\eta\cos\varphi)^2 + (\cosh\xi\cos\eta\sin\varphi)^2 \qquad (4.20)$$

and as $r_{equ} = \cosh\xi$ we can deduce that η_{max} must be limited to:-

$$\eta_{max} = \cos^{-1}\left(\frac{r_{core}}{r_{equ}}\right) \qquad (4.21)$$

4.2 Simulating a partial disk

We can, again for a very oblate spheroid, calculate the emission from a annular disk segment (defined by "start" and "end" values of φ) by ignoring contributions to the integral if φ obtained from:

$$\varphi = \sin^{-1}\left(\frac{K}{\cos\eta}\right) \qquad (4.23)$$

Lies outside these values.

5.0 Convolution of Two Distributions

Given a histogram starting distribution vector (V_0) with known (not necessarily uniform) bin widths ($\Delta\lambda_i$) we can apply a second spreading distribution to yield the resultant distribution vector (V_1) via the matrix operation:-

$$MV_0 = V_1 \tag{5.1}$$

where $m_{ij} = D(\lambda_j - \lambda_i)\left(\frac{\Delta\lambda_j}{\Delta\lambda_i}\right)$ and D is the second distribution function.

6.0 Saturation Effects in Absorption Lines

Capture and emission processes between two atomic levels with principle quantum numbers i and j (i > j) are governed by the Einstein coefficients which relate the relevant atomic level populations via the equation (see the Appendix):-

$$\frac{N_i}{N_j} = \frac{B_{ji}\frac{\lambda^2}{c}\rho(T,\lambda_{ij})}{A_{ij}[1+P(T,\lambda_{ij})]} \tag{6.1}$$

where:-

- N_i units m^{-3}, is the number density of hydrogen atoms with an electron on the ith energy level
- $\rho(T,\lambda_{ij})$ units J m^{-4} , is the Planck distribution photon energy density per unit volume per unit wavelength at temperature T and transition wavelength λ_{ij}.
- A_{ij} with units s^{-1}, is the Einstein coefficient for spontaneous photon emission from the n=i to n=j level electron transition (i>j).
- B_{ij} units m^3 J^{-1} s^{-2}, is the Einstein coefficient for stimulated emission from the n=i to n=j level electron transition.
- B_{ji} units m^3 J^{-1} s^{-2}, is the Einstein coefficient for photon capture resulting in the n=j to n=i level electron transition.

and we have defined a dimensionless function:-

$$P(T,\lambda_{ij}) \equiv \frac{1}{exp\left(\frac{hc}{kT\lambda_{ij}}\right)-1} \tag{6.2}$$

The Einstein coefficients are fundamental properties of particular transitions within a particular atomic species and are independent of external factors.

Using the definition (6.2) we can express $\rho(T,\lambda_{ij})$ as:-

$$\rho(T,\lambda_{ij}) = \frac{8\pi hc}{\lambda_{ij}^5}P(T,\lambda_{ij}) \tag{6.3}$$

and therefore we can write:-

$$\frac{N_i}{N_j} = \frac{8\pi hB_{ji}P(T,\lambda_{ij})}{\lambda_{ij}^3 A_{ij}[1+P(T,\lambda_{ij})]} = \frac{8\pi hB_{ji}}{\lambda_{ij}^3 A_{ij}}exp\left(-\frac{hc}{kT\lambda_{ij}}\right) \tag{6.4}$$

All constants introduced in equations (6.2) through to (6.4) take their usual physical meanings.

We now define a photon "capture cross-section" σ_{ji} via:-

$$\sigma_{ji} \equiv \sigma\frac{hB_{ji}}{\lambda_{ij}A_{ij}} \tag{6.5}$$

where σ is a dimensionless constant into which all appropriate numeric constants are bundled. The reasons behind this particular definition are discussed in section 4 but it should be noted here that with this definition the capture cross-section has the required units of area. Using equation (6.5) equation (6.4) can now be re-arranged to yield:-

$$\sigma_{ji} = \frac{\sigma\lambda_{ij}^2}{8\pi} exp\left(\frac{hc}{kT\lambda_{ij}}\right)\left[\frac{N_i}{N_j}\right] \quad (6.6)$$

Boltzmann statistics allows us to express the ratio $\frac{N_i}{N_j}$ as:-

$$\frac{N_i}{N_j} = \frac{g_i}{g_j} exp\left(-\frac{hc}{kT\lambda_{ij}}\right) \quad (6.7)$$

Where $g_i = 2i^2$ represents the statistical weight of the ith atomic level. Substituting in equation (6.6) we obtain:-

$$\sigma_{ji} = \frac{\sigma\lambda_{ij}^2}{8\pi}\frac{g_i}{g_j} \quad (6.8)$$

Equation (6.8) achieves our first analysis aim i.e. to determine photon capture cross-sections from an assumed state of thermal equilibrium.

To calculate the emerging spectrum we note that as photons of a given wavelength are indistinguishable quantum particles, we cannot determine the route that any particular photon takes through a spectrosphere. Therefore any configuration of routes that satisfy the boundary conditions is a possible configuration and must in principle be indistinguishable from other possible configurations.

One configuration in particular is very amenable to calculation, and that one assumes that only the photons that suffer no scattering on their journey across the spectrosphere will emerge. All photons that suffer a scattering event are assumed to be recycled into the continuum to maintain thermal equilibrium and are therefore not emitted from the upper surface. This is a possible configuration, as a photon that is not scattered will certainly emerge. It also satisfies all boundary conditions so any process that would deviate from the observed result, by the principle of detailed balance, must be compensated for by an equivalent inverse process on average.

So to proceed consider a stream of photons at temperature T crossing the spectrosphere along our line of sight, the number of captures in wavelength range $d\lambda$ at wavelength λ occurring within a small distance dx is given by $\sigma_{ji}N_j(T,\lambda)d\lambda dx$ where $N_j(T,\lambda)$ is the effective, box rest frame, number density of atoms in a state of motion and ionisation able to absorb a photon at wavelength λ per unit wavelength.

There are two points we need to note:-

1. My software[3] re-samples all spectra such that $d\lambda$ is constant over the spectral range, this allows measured spectra (intensity per pixel) to be directly compared to distributions (intensity m^{-2} λ^{-1}) as they are proportional with the constant of proportionality $d\lambda$.
2. In the atom rest frame, all captures occur at wavelength λ_{ij} hence the assumed constant capture cross-section.

Now, $N_j(T,\lambda)$ has units of m^{-3} λ^{-1} therefore we have:-

$$N_j = \int N_j(T,\lambda)d\lambda \qquad (6.9)$$

The change in normalised radiation flux at wavelength λ in range $d\lambda$ as a function of x can be expressed as:-

$$\mathrm{d}\left[\frac{f(x,T,\lambda)}{\mu(T,\lambda_{ij})}\right] = -\sigma_{ji}\left[\frac{f(x,T,\lambda)}{\mu(T_p,\lambda_{ij})}\right]N_j(T,\lambda)d\lambda dx \qquad (6.10)$$

Where $f(0,T,\lambda) \equiv \mu(T,\lambda_{ij})$, i.e. the continuum is assumed constant over the width of the line . Equation (6.10) can be integrated to give the, normalised continuum, emerging radiation flux within the range $d\lambda$ which we will denote as $L_{ij}(T,\lambda)$:-

$$L_{ij}(T,\lambda) \equiv \frac{f(t,T,\lambda)}{\mu(T,\lambda_{ij})} = exp\{-\sigma_{ji}N_j(T,\lambda)d\lambda\, t\} \qquad (6.11)$$

Substituting from equation (6.8) we obtain:-

$$L_{ij}(T,\lambda) = exp\{-e_{ij}(T,\lambda)d\lambda\, N_j t\} \qquad (6.12)$$

where t is the thickness of the absorbing layer and we have defined an "equivalent emission line" profile that mirrors all capture processes (principle of detailed balance) via:-

$$e_{ij}(T,\lambda) \equiv \frac{\sigma\lambda_{ij}^2}{8\pi}\frac{g_i}{g_j}\frac{N_j(T,\lambda)}{N_j} \qquad (6.13)$$

If we also define $L_{ij}(T,\lambda_{ij}) \equiv L_{ij}$ and $e_{ij}(T,\lambda_{ij}) \equiv e_{ij}$, when $\lambda = \lambda_{ij}$ we can deduce from (6.12) that:-

$$N_j t = \frac{-ln(L_{ij})}{e_{ij}d\lambda} \qquad (6.14)$$

Substituting from (6.14) into (6.12) yields:-

$$L_{ij}(T,\lambda) = exp\left\{\frac{e_{ij}(T,\lambda)}{e_{ij}}\ln[L_{ij}]\right\} = exp\{E_{ij}(T,\lambda)\ln[L_{ij}]\} \qquad (6.15)$$

Where $E_{ij}(T,\lambda) \equiv \frac{e_{ij}(T,\lambda)}{e_{ij}}$ is the normalised, $E_{ij}(T,\lambda_{ij}) \equiv 1$, equivalent emission line profile. Equation (6.15) can be inverted to yield:-

$$E_{ij}(T,\lambda) \equiv \frac{\ln[L_{ij}(T,\lambda)]}{\ln[L_{ij}]} \qquad (6.16)$$

When analysing absorption lines from a star, equation (6.16) is used to generate a normalised "equivalent emission line" $E_{ij}(T,\lambda)$ from the measured absorption line $L_{ij}(T,\lambda)$. This emission line contains all the information regarding the dynamics of the stellar spectrosphere. We can use $E_{ij}(T,\lambda)$ as the target for determining physical properties of the star by simulating emission lines for a given temperature, pressure and state of rotation and comparing them to $L_{ij}(T,\lambda)$.

Having thus modelled a star we are able to compute the profile of any emission line in a particular series however our aim is to predict absorption line profiles so there is still work to be done.

Before moving on to the prediction of absorption line profiles we explain the significance of the "equivalent emission line" a little more. The absorption profile $L_{ij}(T,\lambda)$ and the emission profile $E_{ij}(T,\lambda)$ represent the mathematical embodiment of Kirchoff's spectral laws i.e:-

- Absorption lines $L_{ij}(T,\lambda)$ are seen when a continuous spectrum light source is viewed through an intervening gas cloud.
- Emission lines $E_{ij}(T,\lambda)$ are seen when light from a continuous spectrum light source is scattered into our line of sight by a gas cloud adjacent to the source.

Therefore for a given gas cloud illuminated by a given light source one of the spectral lines $L_{ij}(T,\lambda)$ or $E_{ij}(T,\lambda)$ will be observed according to the observers viewpoint.

6.1 Relation between saturation effects in lines from the same element

For a given n=j base level line series, our task here is to determine the expected shape of the mth absorption line after modelling a measured *i* th absorption line. From equation (2.16) we know that**:-**

$$N_j t = \frac{-\ln[L_{ij}]}{e_{ij}d\lambda_i} = \frac{-\ln[L_{mj}]}{e_{mj}d\lambda_m} \tag{6.17}$$

From which we can deduce:-

$$L_{mj} = [L_{ij}]^{\frac{e_{mj}d\lambda_m}{e_{ij}d\lambda_i}} \tag{6.18}$$

We now need to determine a value for the ratio $\frac{e_{mj}}{e_{ij}}$.

We can calculate the area a_{ij} (the "equivalent width" units m) under our simulated normalised emission lines using equation (6.13):-

$$a_{ij} = \frac{\sigma}{8\pi}\frac{\lambda_{ij}^2}{e_{ij}}\frac{g_i}{g_j} \tag{6.19}$$

therefore:-

$$e_{ij} = \frac{\sigma}{8\pi}\frac{\lambda_{ij}^2}{a_{ij}}\frac{g_i}{g_j} \tag{6.20}$$

and so finally we arrive at:-

$$L_{mj} = [L_{ij}]^{\left(\frac{\lambda_{mj}}{\lambda_{ij}}\right)^2 \frac{a_{ij}}{a_{mj}}\frac{g_m}{g_i}\frac{d\lambda_m}{d\lambda_i}} \tag{6.21}$$

Equation (6.21) defines the required relationship between L_{mj} and L_{ij}.

Having obtained L_{mj} and computed the $E_{mj}(\lambda)$ emission line, from the known stellar model parameters, equation (6.15) can now be used to predict the required absorption line $L_{mj}(T,\lambda)$.

Another interesting consequence of equation (6.19) is that we can substitute into (6.14) to obtain:-

$$\sigma N_j t = \frac{-8\pi g_j a_{ij} ln(L_{ij})}{g_i \lambda_{ij}^2 d\lambda_i} \tag{6.22}$$

which would give, once a value for the constant σ is determined, the number of atoms that are in the required state to absorb a λ_{ij} photon per unit surface area of the star i.e. the column density of atoms in principle quantum state j.

I will make the identification:-

$$\sigma = 4\pi\alpha^3 \tag{6.23}$$

where α is the "fine structure constant", giving:-

$$N_j t = \frac{-2 g_j a_{ij} ln(L_{ij})}{\alpha^3 g_i \lambda_{ij}^2 d\lambda_i} \tag{6.24}$$

Equation (6.24) allows, for a given set of stellar model parameters, $N_j t$ to be calculated.

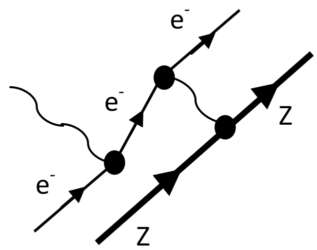

Figure 2: Feynman diagram for atomic photon absorption

The reason for the assignment (6.23) is illustrated in figure 2, where the Feynman diagram representing photon absorption is depicted. It should be noted that absorption of a single photon by a single electron is kinematically forbidden, as energy and momentum cannot both be conserved, therefore figure 2 shows the simplest atomic absorption process. It involves a photon exchange with the nucleus as described by this 3 vertex Feynman diagram. In this diagram the atomic nucleus is labelled Z, the electron is labelled e^- and the photons are indicated by wiggly lines. It can be seen that a nucleus, electron and photon enter from the left whilst just the nucleus and an excited electron exit to the right. Three vertices imply that the differential interaction cross section includes a dimensionless factor of $\alpha^3 = \left(\frac{1}{137}\right)^3 = 3.89e^{-7}$ where α is the fine structure constant. Integration over all angles introduces a further factor of 4π yielding equation (6.23)

7.0 Relationship Between Mean-Free-Path and Pressure

We know that the concentration of atoms is the reciprocal of the average volume swept out by an atom between collisions i.e:-

$$N = \frac{1}{\pi r_x^2 l} \tag{7.1}$$

where N is the total number density of neutral and ionised atoms in a photosphere, l is the mean free path (determined from modelling a spectral line) and the "effective" collision cross sectional area $= \pi r_x^2$. Now by splitting the collision cross-section into a neutral and ionized atom part we can write:-

$$r_x^2 = \frac{N_I r_n^2 + N_{II} r_i^2}{N} \tag{7.2}$$

where r_n is the "effective" radius of a neutral atom, r_i is the "effective" radius of an ionized atom, N_I and N_{II} are respectively the neutral and ionised hydrogen concentrations. We can calculate a weighted average classical radius for a neutral atom at any given temperature but, as quantum mechanically an electron wave function has a significant value a considerable distance beyond the classical radius a multiplier (q_m) has been introduced i.e:-

$$Nr_x^2 = N_I(q_m r_n)^2 + N_{II} r_i^2 \tag{7.3}$$

We can calculate a weighted average classical radius for a neutral atom at any given temperature but to calculate r_i we need some more physics in the form of "Rutherford Scattering" theory which states that:-

$$\sigma(\theta > \theta_0) = \frac{\pi}{4}\left[\frac{z_1 z_2 e^2}{4\pi\epsilon_0 E}\cot\left(\frac{\theta}{2}\right)\right]^2 \tag{7.4}$$

where:-

- $\sigma(\theta > \theta_0)$ represents the cross-section for scattering at all angles θ greater than θ_0
- z_1 and z_2 are the atomic number of the scattered and scattering nucleus
- e is the electronic charge
- ϵ_0 is the permittivity of free space
- E represents the kinetic energy of the system

We will assume that:-

- $z_1 = 1.085$ which is the average charge of scattered nuclei (H 91.5%, He 8.5% by number)
- $z_2 = 1.0$ i.e. Hydrogen is the scattering atom
- $E = \frac{3}{2}kT$ which is the average particle energy as a function of temperature T

We also need to define what we mean by a "collision" in the context of determining a contribution of ionised atoms to the total spectrosphere pressure. As the electromagnetic force is infinite in range the total scattering cross section $\sigma(\theta > 0)$ is infinite. We will regard a collision to have occurred for all scattering angles $\theta > \frac{\pi}{2}$ and therefore set $\theta_0 = \frac{\pi}{2}$, in this case the scattered particle has a component of its outgoing velocity in the opposite direction to its incoming velocity which seems a sensible definition for a collision. Equation (7.4) can thus be re-cast as:-

$$r_i = \frac{z_1 z_2 e^2}{12\pi\epsilon_0 kT} \tag{7.5}$$

For the Sun equation where $T \approx 6000K$ equation (7.5) yields the value $r_i = 1.005e - 9$m whilst the corresponding value for r_x is $r_x = q_m r_b = 2.65e - 10$ where we have set $q_m = 5$ and r_b is the Bohr radius of the Hydrogen atom (= 5.29e-11m). Combining this with the known particle density of 2.531e-4 kg/m^3 we can deduce that $\Lambda \approx 1/80$.

We can now go on to determine the photospheric pressure, the perfect gas law states that:-

$$PV = nRT \tag{7.6}$$

where *P* is pressure, *V* is volume, *T* is absolute temperature, n is the number of moles of the particles, *R* (= 8.31441) is the molar gas constant therefore:-

$$P = \frac{n}{V}RT \equiv n_v RT \tag{7.7}$$

where n_v is the number of moles of the particles per unit volume, but *N* is the number of particles per unit volume (that we now know) so:-

$$P = \frac{N}{N_A}RT \tag{7.8}$$

where N_A is Avogadro's number (= 6.022045e23). An alternative way of writing the same equation is:-

$$P = NkT \tag{7.9}$$

Where *k* is Boltzmann's constant (=1.380662e-23).

When applied to the Sun we obtain:-

$$P_{sun} = 0.107Bar \tag{7.10}$$

which compares well with the published data shown in table 1. This agreement critically depends on and therefore lends support to the assignment represented by equation (3.2). Also note that it is

Solar Photosphere as a Function of Depth

Depth (km)	% Light from this Depth	Temperature (K)	Pressure (bars)
0	99.5	4465	6.8×10^{-3}
100	97	4780	1.7×10^{-2}
200	89	5180	3.9×10^{-2}
250	80	5455	5.8×10^{-2}
300	64	5840	8.3×10^{-2}
350	37	6420	1.2×10^{-1}
375	18	6910	1.4×10^{-1}
400	4	7610	1.6×10^{-1}

Source: Fraknoi, Morrison, and Wolf, *Voyages through the Universe*

Table 1: Published data on the Solar photosphere

independent of the other assignment represented by equation (6.23).

8.0 Other interesting photosphere properties

We can approximate N_j as:-

$$N_j \approx N_I \exp\left[-\frac{hc}{kT\lambda_{12}}\right] \tag{8.1}$$

where N_I is the number of neutral atoms per unit volume and λ_{12} is the Lyman α wavelength = 1216 A, therefore (6.24) becomes:-

$$t = \frac{-2g_j \mathrm{a_{ij}} \ln[L_{ij}]}{\alpha^3 \lambda_{ij}^2 d\lambda_i g_i N_I exp\left[-\frac{hc}{kT\lambda_{12}}\right]} \qquad (8.2)$$

For the Sun's spectrosphere (8.2) yields t_{sun}~ 4.411e5 m which is very close to the generally accepted value of 400km. This agreement critically depends on and therefore lends support to the assignment represented by equation (6.23).