# A Spectroscopic Study of the Blue component of Albireo 

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#### Abstract

This paper describes an investigation into what can be deduced about the physical properties of the blue component of the Albireo double star system from both low ( 150 lines $/ \mathrm{mm}$ ) and high ( 2400 lines $/ \mathrm{mm}$ ) resolution spectra, based on the simple model that the star is a rotating uniformly emitting oblate spheroid with a photosphere that is a single layer in thermal equilibrium.

The blue component of Albireo is an interesting target in that it exhibits emission at both $\mathrm{H}_{\alpha}$ and, to a lesser extent, $\mathrm{H}_{\beta}$ wavelengths, this emission is believed to originate from an equatorial decretion disk spun off from the star. The aim of this work was to split the observed high resolution spectra into an absorption component, from the star, and an emission component, from the disk. To achieve this aim the continuum spectrum was modelled as a "black body" to obtain an effective photosphere temperature and the $\mathrm{H}_{y}$ absorption line was analysed to obtain values for the star's model parameters. These results were then used to predict the expected absorption at $\mathrm{H}_{\alpha}$ and $\mathrm{H}_{\beta}$ wavelengths. Measured $\mathrm{H}_{\alpha}$ and $\mathrm{H}_{\beta}$ lines were then divided by their expected absorption lines to restore the continuum which was then subtracted to reveal the pure disk emission for future analysis.


## 1. Introduction

The blue component of the Albireo ( $\beta \mathrm{Cyg}$ ) double star system (Albireo B), see figure 1, is classed as a B8Ve star i.e. a hot (B8) main sequence (V) star that displays evidence of emission lines (e) in its spectrum.


Figure 1: Spectra from the Albireo double star plus, inset, an image of the pair.

A "normal" star would be expected to display only absorption lines, superimposed on the top of the thermal continuum. This absorption is due to continuum photons, at characteristic wavelengths, being absorbed by atoms and then re-emitted in random directions and "thermalised" (returned to the continuum) as they pass through the star's atmosphere. The region of a star's atmosphere that is mainly responsible for forming the observed spectrum is called the photosphere.

The emission component from Albireo B is believed to originate from an equatorial decretion disk spun off from the rapidly rotating star and this emission is then superimposed on top of the normal absorption lines from the star itself.

The aim of this work was to separate these two spectral line components, using a simple physical stellar model, and in the process deduce properties of the star and disk.

The stellar model used was that of an oblate spheroid solid body rotating uniformly with a photosphere that is a single layer in thermal equilibrium.

Using this model an effective "black body" temperature can be deduced from low resolution (150 lines $/ \mathrm{mm}$ ) spectra provided proper calibration is performed to correct the continuum spectrum for instrument response and atmospheric absorption. High resolution ( 2400 lines $/ \mathrm{mm}$ ) investigations of individual line shapes can then be used to determine other model parameters for example, the half width of the pressure induced Lorentzian distribution, the rotation surface velocity and angle of view.

The key to separating absorption and emission line components for Albireo B was the fact that there was no evidence of emission at the $\mathrm{H}_{\gamma}$ wavelength and therefore this line could be modelled to establish the physical properties of the star itself. The expected absorption at $\mathrm{H}_{\alpha}$ and $\mathrm{H}_{\beta}$ wavelengths could then be computed and the pure disk emission lines deduced by dividing measured $\mathrm{H}_{\alpha}$ and $\mathrm{H}_{\beta}$ lines by the expected absorption line profiles and subtraction of the continuum.

Once separated the disk emission can be further analysed to gain some insight into the disk properties
(e.g. orientation) and this will be the subject of a subsequent paper.

A suite of custom computer programs was written to perform the required computations and this software is freely available from the author.

## 2. Measurements

All spectra were captured using an Atik 314L+ camera in conjunction with a Shelyak LHiresIII spectrograph mounted on a Meade 12" LX200ACF telescope and initial processing was performed using the commercially available RSpec ${ }^{1}$ software package.

### 2.1 Low Resolution Spectra

Figure 2 shows a low resolution ( 150 lines/mm) spectrum of Albireo $B$, this spectrum was fully calibrated for instrument response and atmospheric absorption using a library reference spectrum. In the figure the Hydrogen $\alpha, \beta$ and $\gamma$ line positions have been indicated.

Emission at $\mathrm{H}_{\alpha}$ is immediately apparent whilst, at low resolution, both the $\mathrm{H}_{\beta}$ and $\mathrm{H}_{\gamma}$ lines seem to be normal absorption lines.

Low resolution data can be used to obtain an estimate for the effective temperature of a star. It is simply necessary to divide the spectrum by a "Planck wavelength curve" defined by the equation:-

$$
P(T, \lambda)=\frac{2 \pi h c}{\lambda^{5}} \frac{1}{\left(\exp \left(\frac{h c}{\lambda k T}\right)-1\right)}
$$

Where all symbols take their usual physical meanings. The temperature T is adjusted until the result of the division is the closest to a horizontal straight line, as measured by a minimum root mean square deviation from the mean.


Figure 2: $150 \mathrm{I} / \mathrm{mm}$ spectrum of $\begin{gathered}\text { Angstroms } \\ \text { Albireo } B\end{gathered}$


Figure 3: Flattened line spectrum of Albireo B.

Custom software has been written to calculate this optimal temperature automatically, for Albireo B the result is an estimate of 17625 K .

Figure 3 shows the flattened spectrum of Albireo B after division by the appropriate Planck curve.

### 2.2 High Resolution Spectra

High resolution ( 2400 line $/ \mathrm{mm}$ ) spectra taken at $\mathrm{H}_{\gamma}, \mathrm{H}_{\beta}$ and $\mathrm{H}_{\alpha}$ wavelengths are shown in figures 4,5 and 6 respectively.


Figure 4: $\mathbf{2 4 0 0}$ l/mm spectrum of Albireo $\mathbf{B}$ at $\mathbf{H}_{\gamma}$
The $\mathrm{H}_{\gamma}$ line appears to be the result of pure absorption whilst the $\mathrm{H}_{\beta}$ line displays a curious "feature" at its centre and the $\mathrm{H}_{\alpha}$ line appears to be largely an emission line with a double peak characteristic of the effects of rotation.


Figure 5: $\mathbf{2 4 0 0}$ I/mm spectrum of Albireo $B$ at $H_{\beta}$


Figure 6: $\mathbf{2 4 0 0}$ I/mm spectrum of Albireo $B$ at $H_{\alpha}$

## $2.3 \mathbf{H}_{\gamma}$ line analysis

For the $\mathrm{H}_{\gamma}$ absorption line, using the custom software, the continuum gradient was removed and the central wavelength determined, based on equal areas each side of centre. The intensity at the absorption line centre was found to be 0.455 , this value will be required when predicting other absorption lines in the hydrogen Balmer series. The measured absorption line profile $A(T, \lambda)$ was first converted to an equivalent normalized emission line $E(T, \lambda)$ using the equation:-

$$
\begin{equation*}
E\left(T, \lambda_{\gamma}\right) \equiv \frac{\ln [A(T, \lambda)]}{\ln [A 0]} \tag{2.1}
\end{equation*}
$$

where $A 0=A\left(T, \lambda_{\gamma}\right)$ is the absorption line intensity at the line centre.

The bulk of the theory behind the modelling was presented in a paper entitled "Validating a Simple Analysis Model of Spectral Line Profiles Using Solar Reflection Spectra from Jupiter's Moon Europa" available from the author's website ${ }^{2}$ in the file Europa.pdf ${ }^{3}$. The additional theory required to
model stellar rotational effects is presented in appendix A at the end of this document.

The software generates an "effective emission line" from a measured absorption spectrum, this line represents the pure dynamics of the atoms in the photosphere and is the target for simulations. Input parameters for the simulation are:-

- AMass: I assume the effective atomic mass of the atoms is $1.255(91.5 \% \mathrm{H}$ and $8.5 \% \mathrm{He}$ atoms by number) a different figure can be input to the simulations if preferred.
- T : the photosphere temperature in Kelvin
- LHW: the Lorentzian pressure distribution half width in Angstrom.
- vrot: the equatorial surface velocity of the star as a proportion of c .
- Ob : the Oblateness of the star i.e. equatorial radius divided by polar radius ( $\mathrm{Ob} \geq 1$ ).
- Theta: the viewing angle in radians above the stellar equator.

The next step is to run the model whilst adjusting the model parameters until a good fit is obtained between the modelled line and the effective emission line. In this process absorption lines at other wavelengths in the Balmer series can be predicted and compared with experiment to help separate out the different possible combinations of input parameters. Figure 7 shows how combinations of rotational velocity and the pressure Lorentzian Half Width (LHW) affect the emission line shape at $H_{\gamma}$ for similar overall line halfwidth. Figure 8 shows how these same combinations affect predicted absorption lines at $H_{\alpha}$.

Figure 7: Rotation and Pressure Effect on $\mathrm{H}_{\boldsymbol{\gamma}}$ Line Profiles


The equivalent emission line and the final result of the modelling process are displayed in Figure 9. The model parameters that will be used in the rest of this
paper are displayed as labels in this figure. Also appearing as labels in this figure are the following quantities, determined by the software, that are required for predicting the absorption line profile at $H_{\alpha}$ and $H_{\beta}$ :-

- L0: the peak wavelength.
- dl0: the bin width of the spectrum, the software re-samples the initial data to a uniform bin width.
- A 0 : the amplitude of the absorption line at the peak wavelength
- Wequiv: the equivalent width of the model emission line.


The final optimised input parameters of the model are:-

- $T=17625 \mathrm{~K}$
- $\mathrm{LHW}=5.0 \mathrm{~A}$
- $\mathrm{V}_{\text {rot }}=0.001 \mathrm{c}$
- $\mathrm{Ob}=1.5$
- $\quad$ Theta $=0.25 \mathrm{pi}$

Note that the value of Theta was chosen to be 0.25 pi ( 45 degrees) because the star's equator, as defined by the decreation disk, must be angled so that the disk is significantly "open" but as we also see a double peak in the disk emission we cannot be looking down the rotational pole. There is too little data to expect the
angle of view to be constrained much more than this rather crude estimate. The estimate of oblateness is similarly in-exact.
The calculated emission line can be converted back into an absorption line, by the inverse of equation (2.1) and compared to the measured absorption line. The result is shown in figure 10 where it can be seen that the fit between model and measurement is really quite good.


In performing this transformation a calculation of the column density of hydrogen atoms in state $i=$ 2 is made and stored as an Rspec label, its value $(\mathrm{N} 2 \mathrm{t})$ is displayed in figure 10 along with the calculated photon capture cross-section $\sigma$. At this point the software also calculates and stores as labels estimates of the Pressure and thickness of AlbireoB's photosphere but this process will need more explanation and so will be discussed in section 2.5.

## 2.4 $H_{\beta}$ and $H_{\alpha}$ line synthesis

The custom software can now be used to compute the expected absorption lines at $\mathrm{H}_{\beta}$ and $\mathrm{H}_{\alpha}$ wavelengths. The results of these calculations are displayed in figures 11 and 12 respectively.


Having computed the expected absorption line profiles, the measured line profiles can now be divided by the corresponding calculated profiles to restore the continuum illumination which can then be subtracted to reveal the pure emission lines
originating from the decretion disk orbiting the star. The result of this operation for the $\mathrm{H}_{\beta}$ and $\mathrm{H}_{\alpha}$ lines are displayed in figures 13 and 14 respectively.


These disk line profiles will be further analysed in a subsequent document.

### 2.5 Photosphere Pressure and Thickness

As stated in section 2.3, when generating the absorption profile corresponding to the modelled equivalent emission line, the software performs a calculation that allows an estimate to be made of the target star's photosphere pressure and thickness as a function of an "impact parameter" $\rho$. Figure 15 plots the pressure and thickness of Albireo B's photosphere as a function of this impact parameter.

When analysing the Solar photosphere ${ }^{3}$ it was possible to deduce an impact parameter value appropriate to the Sun's known photosphere properties however, this value will not be appropriate
for all stars as it is expected to depend on the types of impacts. In cool stars, such as the Sun, neutral on neutral atom impacts dominate i.e. dipole-dipole electric field interactions. In the case of the hottest stars, ionised on neutral atom impacts will dominate i.e. monopole-dipole electric field interactions which are expected to be of "longer range" i.e. a larger value of $\rho$.


As verified in the study of the Solar photosphere ${ }^{3}$, we can use known values of stellar surface gravity to estimate photosphere pressure as a function of thickness given the value for the column density of neutral atoms in the Balmer ground state calculated by the software and displayed in figure 10 .
A plot of these independent functions of pressure vs thickness is displayed in figure 16 given that Albireo B's surface gravity is $100 \mathrm{~ms}^{=2}$. In this figure we have plotted two surface gravity curves, the green curve assumes the pressure at the base of the photosphere whilst the red curve assumes half this base pressure as a better equivalent to the impact parameter derived pressure ${ }^{3}$.


The intersection of the blue and red curves in figure 16 thus yield an estimate of approximately 2450 km and 3.6e-4 Bar for the thickness and pressure of Albireo B's photosphere.

## 3. Discussion

The successful flattening of the low resolution spectrum, on division by a $\mathrm{T}=17625$ Planck function (Figure 3), is an indication that the star is well approximated by a "black body" over the range of visible wavelengths. Therefore we might also expect that the high resolution modelling would be successful, as indeed it seems to be.

There will undoubtedly be stars that do not fit with the simple model used here but, in those cases, it may be possible to speculate upon the nature of the departure from the model.

When modelling Albireo B's photosphere there is not enough data to determine all the model input parameters to a high accuracy. In particular Albireo B's oblateness (1.5) was just chosen to be larger than Altair's known value (1.245). Also the viewing angle above the equator 'Theta' was crudely estimated from the fact that the stars decreation disk was open and exhibiting a double peak typical of rotation. However it may be possible to make some progress in splitting candidate parameter sets by comparing predicted and measured lines for multiple Balmer series line members when they are all pure absorption lines.

The goal of separating the photosphere and disk components of the spectrum seems to have been achieved satisfactorily and further analysis of the disk emission profiles will be presented in a follow-up paper.

It was possible to estimate the values of Albireo B's photosphere pressure and thickness by combining the two methods of calculating photosphere pressure as a function of thickness. Those two methods being the impact parameter and surface gravity methods. By studying stars with a range of temperatures it may
prove possible to experimentally derive the impact parameter as a function of stellar temperature.

## 4. Conclusions

A spectroscopic study of the blue component of the Albireo double star system has been performed to determine the physical properties of the star and to separate its spectrum from that of its associated equatorial decretion disk. It has been found that:-

- The approximate black body temperature of the Albireo B's photosphere is 17625 K .
- The pressure induced Lorentzian distribution half width is 5 A .
- The maximum surface velocity is 0.001 c .
- The star is being viewed from an intermediate elevation, approximately $0.25 \pi$ rads above its equator.
- The thickness of Albireo B's photosphere is estimated to be 2450 km .
- The pressure of Albireo B's photosphere is estimated to be 3.6e-4 Bar.

In addition software has been written to perform all the required data processing and this software is freely available from the author should others wish to use it in the study of stellar spectra. The software is very easy to use and works best with RSpec software but will work with any spectra represented as two columns of data i.e. wavelength and intensity.

## 5. References

1. www.rspec-astro.com
2. www.thewhightstuff.co.uk
3. www.thewhightstuff.co.uk Astrospectroscopy Projects Europa pdf

## Appendix A: Rotational Spectral Line Broadening for a Uniformly Emitting Oblate Spheroid

Using oblate spheroidal co-ordinates, let a point on the surface of the star have position vector (relative to its centre):-

$$
\begin{equation*}
\underline{r}=a(\cosh \xi \cos \eta \cos \varphi \underline{i}+\cosh \xi \cos \eta \sin \varphi \underline{j}+\sinh \xi \sin \eta \underline{k}) \tag{A.1}
\end{equation*}
$$

$a>0, \xi \geq 0,-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2 \pi$ and the volume integral can be written as:-

$$
V=\iiint h_{\xi} h_{\eta} h_{\varphi} d \xi d \eta d \varphi
$$

With $h_{\xi}=h_{\eta}=a\left(\sinh ^{2} \xi+\sin ^{2} \eta\right)^{1 / 2}$ and $h_{\varphi}=a \cosh \xi \cos \eta$
In this analysis I will assume that the star is rotating about the z axis i.e. $\underline{\omega}=\omega \underline{k}$ and we are observing the star from within the $\underline{i}, \underline{k}$ plane at a angle $\vartheta$ relative to the x axis i.e from the direction:-

$$
\begin{equation*}
\underline{\hat{d}}=\cos \vartheta \underline{i}+\sin \vartheta \underline{k} \quad \text { where } 0 \leq \vartheta \leq \frac{\pi}{2} \tag{A.2}
\end{equation*}
$$

The degree of oblateness is determined by the coordinate $\xi$ together with the constant $a$. Spherical symmetry results if $\xi \gg 1$ and $a \ll 1$ such that $r_{\text {star }}=a \cosh \xi \cong a \sinh \xi$. Disk symmetry results if $\xi=0$ in which case $r_{d i s k}=a$. At intermediate values of $\xi$ the equatorial radius $\left(r_{E}\right)$ and the polar radius $\left(r_{P}\right)$ are related by:-

$$
\begin{equation*}
\frac{r_{E}}{r_{P}}=\operatorname{coth} \xi \tag{A.3}
\end{equation*}
$$

The first task is to determine the unit normal $(\underline{\hat{n}})$ to a elemental spheroidal surface area at position $\underline{r}$. As the coordinate system is orthogonal curvilinear the normal can be calculated from:-

$$
\underline{\hat{n}}=\frac{1}{h_{\xi}} \frac{d \underline{r}}{d \xi}=\frac{1}{\left(\sinh ^{2} \xi+\sin ^{2} \eta\right)^{\frac{1}{2}}}\{\sinh \xi \cos \eta \cos \varphi \underline{i}+\sinh \xi \cos \eta \sin \varphi \underline{j}+\cosh \xi \sin \eta \underline{k}\}
$$

Therefore, assuming uniform intensity emitted per unit area $\left(I_{0}\right)$, we can determine the total received intensity from:-

$$
\begin{equation*}
I=I_{0} \iint \underline{\hat{d}} \cdot \underline{\hat{n}} h_{\eta} h_{\varphi} d \eta d \varphi=\iint I(\eta, \varphi) d \eta d \varphi \tag{A.4}
\end{equation*}
$$

Where

$$
\begin{equation*}
I(\eta, \varphi)=I_{0} a \cosh \xi \cos \eta\{a \sinh \xi \cos \eta \cos \varphi \cos \vartheta+a \cosh \xi \sin \eta \sin \vartheta\} \tag{A.5}
\end{equation*}
$$

We will now choose to set $a=\frac{1}{\cosh \xi}$ as then equation 4.1 becomes:-

$$
\begin{equation*}
\underline{r}=\cos \eta \cos \varphi \underline{i}+\cos \eta \sin \varphi \underline{j}+\tanh \xi \sin \eta \underline{k} \tag{A.6}
\end{equation*}
$$

and we see that, setting $\eta=0$, the equatorial radius $r_{E}=1$ and, setting $\eta= \pm \frac{\pi}{2}$, the polar radius $r_{P}=\tanh \xi$. We will specify the equatorial surface velocity as a fraction of c , this can be achieved by setting $0 \leq \frac{\omega}{c}<1$.
We want to integrate $I(\eta, \varphi)$ along contours of constant line of sight velocity, therefore we need to calculate the velocity at any point from:-

$$
\underline{v}=\underline{\omega} \wedge \underline{r}=\left|\begin{array}{ccc}
\underline{i} & \underline{j} & \underline{k}  \tag{A.7}\\
0 & 0 & \omega \\
\cos \eta \cos \varphi & \cos \eta \sin \varphi & \tanh \xi \sin \eta
\end{array}\right|
$$

From which we obtain:-

$$
\begin{equation*}
\underline{v}=-\omega(\cos \eta \sin \varphi \underline{i}+\cos \eta \cos \varphi \underline{j}) \tag{A.8}
\end{equation*}
$$

Therefore the line of sight velocity is:

$$
\begin{equation*}
v=\underline{v} \cdot \underline{\hat{d}}=-\omega \cos \eta \sin \varphi \cos \vartheta=\frac{\Delta \lambda c}{\lambda_{0}} \quad \text { (Doppler shift) } \tag{A.9}
\end{equation*}
$$

For a given $\lambda$ and $\lambda_{0}$ define the constant $K$ as:-

$$
\begin{equation*}
K \equiv \cos \eta \sin \varphi=\frac{-\left(\lambda-\lambda_{0}\right)}{\lambda_{0}\left(\frac{\omega}{c}\right) \cos \vartheta} \tag{A.10}
\end{equation*}
$$

which implies $|K| \leq 1$, rearranging we have $\sin \varphi=\frac{K}{\cos \eta}$ and therefore:-

$$
\begin{equation*}
\cos \varphi= \pm\left[1-\left(\frac{K}{\cos \eta}\right)^{2}\right]^{\frac{1}{2}} \tag{A.11}
\end{equation*}
$$

These last equations define the contour along which we wish to evaluate $I(\eta, \varphi)$ for a given value of $K$ i.e. it relates $\eta$ and $\varphi$ so we can determine $I(\eta, \varphi(\eta)) \equiv I(\eta)$ to be:-

$$
\begin{equation*}
I(\eta)=I_{0} \cos \eta\left\{\tanh \xi \cos \eta\left[1-\left(\frac{K}{\cos \eta}\right)^{2}\right]^{\frac{1}{2}} \cos \vartheta+\sin \eta \sin \vartheta\right\} \tag{A.12}
\end{equation*}
$$

For a given value of K (i.e. given $\Delta \lambda$ ) we can evaluate the following integral to obtain the received intensity at a particular $\Delta \lambda$ :-

$$
\begin{equation*}
I(K)=\int_{\eta_{\min }}^{\eta_{\max }} I(\eta) \frac{d l}{d \eta} d \eta \tag{A.13}
\end{equation*}
$$

where $d l$ is the line element given by:-

$$
\begin{equation*}
\frac{d l}{d \eta}=\left[\left(h_{\eta}\right)^{2}+\left(h_{\varphi} \frac{d \varphi}{d \eta}\right)^{2}\right]^{\frac{1}{2}} \tag{A.14}
\end{equation*}
$$

Differentiating equation A. 10 we have:-

$$
\begin{equation*}
\frac{d \varphi}{d \eta}=-\frac{\sin \eta \sin \varphi}{\cos \eta \cos \varphi} \tag{A.15}
\end{equation*}
$$

therefore:-

$$
\begin{equation*}
\frac{d l}{d \eta}=\left[\frac{\sinh ^{2} \xi+\sin ^{2} \eta}{\cosh ^{2} \xi}+\cos ^{2} \eta\left(\frac{d \varphi}{d \eta}\right)^{2}\right]^{\frac{1}{2}} \tag{A.16}
\end{equation*}
$$

Substituting from equations A. 11 and A. 15 we obtain:-

$$
\begin{equation*}
\frac{d l}{d \eta}=\left[\frac{\sinh ^{2} \xi+\sin ^{2} \eta}{\cosh ^{2} \xi}+\frac{\sin ^{2} \eta\left(\frac{K}{\cos \eta}\right)^{2}}{1-\left(\frac{K}{\cos \eta}\right)^{2}}\right]^{\frac{1}{2}} \tag{A.17}
\end{equation*}
$$

Therefore we can write:-
$I(K)=I_{0} \int_{\eta_{\min }}^{\eta_{\max }} \cos \eta\left[\tanh \xi \cos \eta\left[1-\left(\frac{K}{\cos \eta}\right)^{2}\right]^{\frac{1}{2}} \cos \vartheta+\sin \eta \sin \vartheta\right]\left[\frac{\sinh ^{2} \xi+\sin ^{2} \eta}{\cosh ^{2} \xi}+\frac{\sin ^{2} \eta\left(\frac{K}{\cos \eta}\right)^{2}}{1-\left(\frac{K}{\cos \eta}\right)^{2}}\right]^{\frac{1}{2}} d \eta$
We now need to determine the limits of integration, the limits are reached when the contour hits the visible limb of the star. On the limb we have the condition:-

$$
\underline{\hat{n}} \cdot \underline{\hat{d}}=(\sinh \xi \cos \eta \cos \varphi \underline{i}+\sinh \xi \cos \eta \sin \varphi \underline{j}+\cosh \xi \sin \eta \underline{k}) \cdot(\cos \vartheta \underline{i}+\sin \vartheta \underline{k})=0
$$

which implies:-

$$
\begin{equation*}
\sinh \xi \cos \eta \cos \varphi \cos \vartheta+\cosh \xi \sin \eta \sin \vartheta=0 \tag{A.19}
\end{equation*}
$$

substituting for $\varphi$ and re-arranging we have:-

$$
\begin{equation*}
\cos \eta= \pm \sqrt{\frac{1+\left(\frac{K \tanh \xi}{\tan \vartheta}\right)^{2}}{1+\left(\frac{\tanh \xi}{\tan \vartheta}\right)^{2}}} \tag{A.20}
\end{equation*}
$$

This needs to be solved for the two limits, define $-\frac{\pi}{2} \leq \eta_{1} \leq 0$ and $\frac{\pi}{2} \leq \eta_{2} \leq \frac{3 \pi}{2}$. One limb intersection ( $\eta_{2}$ ) will be in the northern hemisphere and the other in the southern. If, as will generally be the case $\eta_{2}>\frac{\pi}{2}$ (exception $\vartheta=\frac{\pi}{2}$ ), we will need to split the integral such that $I=I_{1}-I_{2}$ where $I_{1}$ is integrated between limits $\eta_{1}$ and $\eta_{\max }$ whilst $I_{2}$ is integrated between limits $\eta_{\max }$ and $\eta_{2}$ where $\eta_{\max }$ is calculated from $K=\cos \eta \sin \varphi$ with $\varphi=\frac{\pi}{2}$ i.e.:-

$$
\begin{equation*}
\eta_{\max }=\cos ^{-1}|K| \tag{A.21}
\end{equation*}
$$

The special case of a sphere is obtained by letting $\xi \rightarrow \infty$ in which case equation A. 18 becomes:-

$$
\begin{equation*}
I(K)=I_{0} \int_{\eta_{\min }}^{\eta_{\max }} \cos \eta\left[\cos \eta\left[1-\left(\frac{K}{\cos \eta}\right)^{2}\right]^{\frac{1}{2}} \cos \vartheta+\sin \eta \sin \vartheta\right]\left[1+\frac{\sin ^{2} \eta\left(\frac{K}{\cos \eta}\right)^{2}}{1-\left(\frac{K}{\cos \eta}\right)^{2}}\right]^{\frac{1}{2}} d \eta \tag{A.22}
\end{equation*}
$$

