1.0 Planck Distribution

Plank's distribution law for the energy flux of radiation emitted from a black body at temperature T per unit area per unit time, as a function of frequency, is:-

$$P(T,\nu) = \frac{2\pi h}{c^2}\frac{\upsilon^3}{exp\left(\frac{h\upsilon}{kT}\right)-1} \tag{1.1}$$

If we integrate (contour integration) this over all frequencies we get the total radiated energy flux U (units W m^{-2}) i.e.

$$U = \frac{2\pi h}{c^2}\int_0^\infty \frac{\upsilon^3}{exp\left(\frac{h\upsilon}{kT}\right)-1}d\upsilon = \frac{2}{15}\frac{\pi^5 k^4}{h^3 c^2}T^4 \equiv \sigma T^4 \tag{1.2}$$

The last step in (1.2) produces the Stefan-Boltzmann law where σ is the Stefan-Boltzmann constant (= 5.67x10^{-8} W m^{-2} K^{-4}). Note that the total energy density

$$E = \frac{8\pi h}{c^3}\int_0^\infty \frac{\upsilon^3}{exp\left(\frac{h\upsilon}{kT}\right)-1}d\upsilon = \frac{8}{15}\frac{\pi^5 k^4}{(hc)^3}T^4 = \frac{4}{c}\sigma T^4 \tag{1.2a}$$

Now, given that frequency and wavelength are related by $\nu = \frac{c}{\lambda}$ which implies:-

$$d\nu = -\frac{c}{\lambda^2}d\lambda \tag{1.3}$$

we can write:-

$$U = \frac{2\pi h}{c}\int_0^\infty \frac{c^3}{\lambda^5\left(exp\left(\frac{hc}{\lambda kT}\right)-1\right)}d\lambda = \frac{2}{15}\frac{\pi^5 k^4}{h^3 c^2}T^4 \tag{1.4}$$

$$E = \frac{8\pi h}{c^2}\int_0^\infty \frac{c^3}{\lambda^5\left(exp\left(\frac{hc}{\lambda kT}\right)-1\right)}d\lambda = \frac{8}{15}\frac{\pi^5 k^4}{(hc)^3}T^4 = \frac{4}{c}\sigma T^4 \tag{1.4a}$$

So, Planck's distribution law for the energy flux of radiation emitted from a black body at temperature T per unit area per unit time, as a function of wavelength, is:-

$$P(T,\lambda) = \frac{2\pi hc}{\lambda^5}\frac{1}{\left(exp\left(\frac{hc}{\lambda kT}\right)-1\right)} \tag{1.5}$$

Note that the functions $P(T,\nu)$ and $P(T,\lambda)$ are different and when plotting the distribution against wavelength rather than frequency we are giving the x axis a non-linear stretch (defined by equation (1.3)), therefore the peak frequency in $P(T,\nu)$ and peak wavelength in $P(T,\lambda)$ are, somewhat counter intuitively, not related by $\nu = \frac{c}{\lambda}$.

In prism/grating spectroscopy we are measuring $P(T,\lambda)$ (presumably a photoelectric effect experiment could directly measure $P(T,\nu)$) so we must use equation (1.5) to generate a black body curve.

We can look for the peak positions in the two distribution functions (ν_p and λ_p) by differentiating equations (1.1) and (1.5) then setting both differentials to zero. For $P(T,\nu)$ we obtain:-

$$\frac{\nu_p exp\left(\frac{h\upsilon_p}{kT}\right)\frac{h}{kT}}{exp\left(\frac{h\upsilon_p}{kT}\right)-1} = 3 \qquad (1.6)$$

whilst for $P(T, \lambda)$ we obtain:-

$$\frac{\frac{c}{\lambda_p} exp\left(\frac{hc}{\lambda_p kT}\right)\frac{h}{kT}}{exp\left(\frac{hc}{\lambda_p kT}\right)-1} = 5 \qquad (1.7)$$

Equation (1.7) can be written as:-

$$\frac{\nu_p exp\left(\frac{h\upsilon_p}{kT}\right)\frac{h}{kT}}{exp\left(\frac{h\upsilon_p}{kT}\right)-1} = 5 \qquad (1.7a)$$

with $\nu_p \equiv \frac{c}{\lambda_p}$. Now equations (1.6) and (1.7a) are identical except for the number on the right hand side and so both cannot be true simultaneously. In an attempt to "square the circle" we could compromise and solve:-

$$\frac{\nu_p exp\left(\frac{h\upsilon_p}{kT}\right)\frac{h}{kT}}{exp\left(\frac{h\upsilon_p}{kT}\right)-1} = 4 \qquad (1.8)$$

but this is unphysical. It's better just to accept that $\nu_p \neq \frac{c}{\lambda_p}$ in which case we need to fit the distribution given by equation (1.5) to our prism/grating derived data and calculate the peak from equation (1.7), note as this is a transcendental equation it needs to be iterated. Dropping the p subscript and letting $a \equiv \frac{hc}{kT}$ then we can define the following functions from equation (1.7):-

$$F(\lambda) \equiv \frac{a exp\left(\frac{a}{\lambda}\right)}{\lambda\left[exp\left(\frac{a}{\lambda}\right)-1\right]} - 5 \qquad (1.9)$$

$$\frac{dF(\lambda)}{d\lambda} \equiv F'(\lambda) = \frac{F(\lambda)+5}{\lambda}\left\{F(\lambda) + 4 - \frac{a}{\lambda}\right\} \qquad (1.10)$$

then, to iterate guess a λ_0 and repeatedly calculate:-

$$\lambda_{n+1} = \lambda_n - \frac{F(\lambda_n)}{F'(\lambda_n)} \qquad (1.11)$$

2.0 Thermal Spectral Line Broadening

The probability of a fluctuation ΔE from the mean energy in a system in thermal equilibrium at absolute temperature T is:-

$$P(\Delta E) = P_0 e^{-\frac{\Delta E}{kT}} \qquad (2.1)$$

As $\Delta E = \frac{1}{2}m\Delta V^2$ this corresponds to an atomic velocity fluctuation (ΔV) probability of:-

$$P(\Delta V) = P_0 e^{-\frac{m\Delta v^2}{2kT}} \qquad (2.2)$$

Which, as $\Delta V = c\frac{\Delta\lambda}{\lambda_0}$, in turn corresponds to a Doppler shift (Δλ) probability of:-

$$P(\Delta\lambda) = P_0 e^{-\frac{mc^2\Delta\lambda^2}{2kT\lambda_0^2}} \tag{2.3}$$

i.e. a Gaussian distribution:-

$$P(\lambda, \lambda_0) = P_0 e^{-\frac{(\lambda-\lambda_0)^2}{2\sigma^2}} \tag{2.4}$$

with $\sigma = \sqrt{\frac{kT}{mc^2}}\lambda_0$ and $P_0 = \frac{1}{o\sqrt{2\pi}}$

FWHM: $P(\Delta\lambda_{\text{FWHM}}) = \frac{P_0}{2}$ i.e.:-

$$\frac{1}{2} = e^{-\frac{\Delta\lambda_{\text{FWHM}}^2}{2\sigma^2}} \tag{2.5}$$

Therefore

$$\text{FWHM} = 2\Delta\lambda_{\text{FWHM}} = \sqrt{8\,ln2}\,\sigma \tag{2.6}$$

3.0 Pressure Spectral Line Broadening

Spectral line widths are affected by pressure, the more frequent atomic collisions are the more a given spectral line will be broadened. This is modelled as damped oscillator ordinary differential which has as its solution the Lorentzian distribution (“Atomic Astrophysics and Spectroscopy “ Anil K. Pradhan and Sultana N. Nahar):-

$$\int_0^\infty L(\omega)d\omega = \frac{1}{\pi}\int_0^\infty \left\{\frac{\left(\frac{\Gamma}{2}\right)}{(\omega-\omega_0)^2+\left(\frac{\Gamma}{2}\right)^2}\right\} d\omega = 1 \tag{3.1}$$

$\Gamma = \gamma + \frac{1}{t_0}$ where γ is a quantum mechanical “damping” factor which can be assumed negligible compared to $\frac{1}{t_0}$ which is the average collision frequency, so we have:-

$$\Gamma = \frac{1}{t_0} \tag{3.2}$$

The book referenced above goes on to deduce:-

$$\frac{\Gamma}{2} = Nv_0(\pi\rho_0)^2 \tag{3.3}$$

where *N* is the number density of atoms, ρ_0 is an impact parameter (units m) and v_0 is the relative mean velocity between impacting particles. For a Maxwellian distribution of velocities we have:-

$$v_0 = 4\left[\frac{kT}{\pi M}\right]^{0.5} \tag{3.4}$$

where *M* is the mass of the identical impacting particles.

Changing variable to wavelength using $\omega = \frac{2\pi c}{\lambda}$ we can deduce:-

$$\int_0^\infty L(\lambda)\,d\lambda \approx \frac{1}{\pi}\int_0^\infty \left\{\frac{\left(\frac{\Gamma'}{2}\right)}{(\lambda-\lambda_0)^2+\left(\frac{\Gamma'}{2}\right)^2}\right\} d\lambda = 1 \qquad (3.5)$$

where:-

$$\frac{\Gamma'}{2} = \frac{\lambda^2\frac{\Gamma}{2}}{4\pi c} \qquad (3.6)$$

And the approximately equal sign $\approx$ occurs in (3.5) as we have approximated the term $\lambda\lambda_0$ to λ_0^2 in the change of variable calculation. This results in a symmetric distribution function and introduces negligible errors if, as is the case, the width of a line is small compared with the wavelength. So we have:-

$$L(\lambda) = \frac{1}{\pi}\left\{\frac{\left(\Gamma'/_2\right)}{(\lambda-\lambda_0)^2+\left(\Gamma'/_2\right)^2}\right\} \qquad (3.7)$$

The half height (wavelength half width) occurs when $(\lambda - \lambda_0) = \pm\,\Gamma'/_2$.

We can substitute from (3.3) into (3.6) to obtain an expression for Γ'. Further if we define a parameter ρ via the equation:-

$$\rho_0 = \frac{2\rho^2}{\lambda}\sqrt{\frac{c}{\pi}} \qquad (3.8)$$

Then we can express the wavelength version of half width equation (3.3) in similar form i.e.:-

$$\frac{\Gamma'}{2} = \frac{\lambda^2}{4\pi c} N v_0 (\pi\rho_0)^2 = N v_0 \rho^4 \qquad (3.9)$$

Notes:-

1. I had initially assumed that ρ_0 was a constant for all spectral lines of a particular atom but the definition, in practice, of what constitutes a “collision” and therefore “collision frequency” could depend on the atomic excitation state (line wavelength). In any case ρ_0 should be independent of stellar temperature as this effect is contained within the parameter v_0 in equation (3.9), so once determined on one star the value should transfer to other stars.
2. ρ has units $\sqrt[4]{m^3 s}$

4.0 Rotational Spectral Line Broadening

4.1 Uniformly Emitting Oblate Spheroid

Using oblate spheroidal co-ordinates, let a point on the surface of the star have position vector (relative to its centre):-

$$\underline{r} = a\left(\cosh\xi\cos\eta\cos\varphi\ \underline{i} + \cosh\xi\cos\eta\sin\varphi\,\underline{j} + \sinh\xi\sin\eta\,\underline{k}\right) \qquad (4.1)$$

$a > 0,\ \ \xi \geq 0,\ \ -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2},\ \ 0 \leq \varphi \leq 2\pi$ and the volume integral can be written as:-

$$V = \iiint h_\xi h_\eta h_\varphi d\xi d\eta d\varphi$$

With $\quad h_\xi = h_\eta = a(\sinh^2 \xi + \sin^2 \eta)^{1/2}$ and $h_\varphi = a \cosh \xi \cos \eta$

In this analysis I will assume that the star is rotating about the z axis i.e. $\underline{\omega} = \omega \underline{k}$ and we are observing the star from within the $\underline{i}, \underline{k}$ plane at a angle ϑ relative to the x axis i.e from the direction:-

$$\underline{\hat{d}} = \cos\vartheta\ \underline{i} + \sin\vartheta\,\underline{k} \quad \text{where } 0 \leq \vartheta \leq \frac{\pi}{2} \tag{4.2}$$

The degree of oblateness is determined by the coordinate ξ together with the constant a. Spherical symmetry results if $\xi \gg 1$ and $a \ll 1$ such that $r_{star} = a\cosh\xi \cong a \sinh\xi$. Disk symmetry results if $\xi = 0$ in which case $r_{disk} = a$. At intermediate values of ξ the equatorial radius (r_E) and the polar radius (r_P) are related by:-

$$\frac{r_E}{r_P} = \coth\xi \tag{4.3}$$

The first task is to determine the unit normal $(\underline{\hat{n}})$ to a elemental spheroidal surface area at position $\underline{r}$. As the co-ordinate system is orthogonal curvilinear the normal can be calculated from:-

$$\underline{\hat{n}} = \frac{1}{h_\xi}\frac{dr}{d\xi} = \frac{1}{(\sinh^2\xi + \sin^2\eta)^{\frac{1}{2}}}\left\{\sinh\xi\cos\eta\cos\varphi\underline{i} + \sinh\xi\cos\eta\sin\varphi\underline{j} + \cosh\xi\sin\eta\underline{k}\right\}$$

Therefore, assuming uniform intensity emitted per unit area (I_0), we can determine the total received intensity from:-

$$I = I_0 \iint \underline{\hat{d}}\,.\,\underline{\hat{n}}\; h_\eta h_\varphi d\eta d\varphi = \iint I(\eta,\varphi)\; d\eta d\varphi \tag{4.4}$$

Where

$$I(\eta,\varphi) = I_0 a\cosh\xi\cos\eta\{a\sinh\xi\cos\eta\cos\varphi\cos\vartheta + a\cosh\xi\sin\eta\sin\vartheta\} \tag{4.5}$$

We will now choose to set $a = \frac{1}{\cosh\xi}$ as then equation (4.1) becomes:-

$$\underline{r} = \cos\eta\cos\varphi\ \ \underline{i} + \cos\eta\sin\varphi\,\underline{j} + \tanh\xi\sin\eta\,\underline{k} \tag{4.6}$$

and we see that, setting $\eta = 0$, the equatorial radius $r_E = 1$ and, setting $\eta = \pm\frac{\pi}{2}$, the polar radius $r_P = \tanh\xi$.

We will specify the equatorial surface velocity as a fraction of c, this can be achieved by setting $0 \leq \frac{\omega}{c} < 1$.

We want to integrate $I(\eta,\varphi)$ along contours of constant line of sight velocity, therefore we need to calculate the velocity at any point from:-

$$\underline{v} = \underline{\omega} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega \\ \cos\eta\cos\varphi & \cos\eta\sin\varphi & \tanh\xi\sin\eta \end{vmatrix} \tag{4.7}$$

From which we obtain:-

$$\underline{v} = -\omega\left(\cos\eta\sin\varphi\,\underline{i} + \cos\eta\cos\varphi\,\underline{j}\right) \tag{4.8}$$

Therefore the line of sight velocity is:-

$$v = \underline{v}.\underline{\hat{d}} = -\omega\cos\eta\sin\varphi\cos\vartheta = \frac{\Delta\lambda c}{\lambda_0} \text{ (Doppler shift)} \tag{4.9}$$

For a given λ and λ_0 define the constant K as:-

$$K \equiv \cos\eta\sin\varphi = \frac{-(\lambda-\lambda_0)}{\lambda_0\left(\frac{\omega}{c}\right)\cos\vartheta} \tag{4.10}$$

which implies $|K| \leq 1$, rearranging we have $\sin\varphi = \frac{K}{\cos\eta}$ and therefore:-

$$\cos\varphi = \pm\left[1 - \left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}} \tag{4.11}$$

These last equations define the contour along which we wish to evaluate $I(\eta,\varphi)$ for a given value of K i.e. it relates η and φ so we can determine $I(\eta,\varphi(\eta)) \equiv I(\eta)$ to be:-

$$I(\eta) = I_0\cos\eta\left\{\tanh\xi\cos\eta\left[1 - \left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}}\cos\vartheta + \sin\eta\sin\vartheta\right\} \tag{4.12}$$

For a given value of K (i.e. given $\Delta\lambda$) we can evaluate the following integral to obtain the received intensity at a particular $\Delta\lambda$:-

$$I(K) = \int_{\eta_{min}}^{\eta_{max}} I(\eta)\frac{dl}{d\eta}\,d\eta \tag{4.13}$$

where dl is the line element given by:-

$$\frac{dl}{d\eta} = \left[\left(h_\eta\right)^2 + \left(h_\varphi\frac{d\varphi}{d\eta}\right)^2\right]^{\frac{1}{2}} \tag{4.14}$$

Differentiating equation 4.10 we have:-

$$\frac{d\varphi}{d\eta} = -\frac{\sin\eta\sin\varphi}{\cos\eta\cos\varphi} \tag{4.15}$$

therefore:-

$$\frac{dl}{d\eta} = \left[\frac{\sinh^2\xi+\sin^2\eta}{\cosh^2\xi} + \cos^2\eta\left(\frac{d\varphi}{d\eta}\right)^2\right]^{\frac{1}{2}} \tag{4.16}$$

Substituting from equations (4.11) and (4.15) we obtain:-

$$\frac{dl}{d\eta} = \left[\frac{\sinh^2\xi+\sin^2\eta}{\cosh^2\xi} + \frac{\sin^2\eta\left(\frac{K}{\cos\eta}\right)^2}{1-\left(\frac{K}{\cos\eta}\right)^2}\right]^{\frac{1}{2}} \tag{4.17}$$

Therefore we can write:-

$I(K) =$

$$I_0 \int_{\eta_{min}}^{\eta_{max}} \cos\eta \left[\tanh\xi \cos\eta \left[1-\left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}} \cos\vartheta + \sin\eta \sin\vartheta\right] \left[\frac{\sinh^2\xi+\sin^2\eta}{\cosh^2\xi} + \frac{\sin^2\eta\left(\frac{K}{\cos\eta}\right)^2}{1-\left(\frac{K}{\cos\eta}\right)^2}\right]^{\frac{1}{2}} d\eta \quad (4.18)$$

We now need to determine the limits of integration, the limits are reached when the contour hits the visible limb of the star. On the limb we have the condition:-

$$\underline{\hat{n}}.\underline{\hat{d}} = \left(\sinh\xi \cos\eta \cos\varphi \;\underline{i} + \sinh\xi \cos\eta \sin\varphi \,\underline{j} + \cosh\xi \sin\eta \,\underline{k}\right).\left(\cos\vartheta \;\underline{i} + \sin\vartheta \,\underline{k}\right) = 0$$

which implies:-

$$\sinh\xi \cos\eta \cos\varphi \cos\vartheta + \cosh\xi \sin\eta \sin\vartheta = 0 \quad (4.19)$$

substituting for φ and re-arranging we have:-

$$\cos\eta = \pm\sqrt{\frac{1+\left(\frac{K\tanh\xi}{\tan\vartheta}\right)^2}{1+\left(\frac{\tanh\xi}{\tan\vartheta}\right)^2}} \quad (4.20)$$

This needs to be solved for the two limits, define $-\frac{\pi}{2} \le \eta_1 \le 0$ and $\frac{\pi}{2} \le \eta_2 \le \frac{3\pi}{2}$. One limb intersection (η_2) will be in the northern hemisphere and the other in the southern. If, as will generally be the case $\eta_2 > \frac{\pi}{2}$ (exception $\vartheta = \frac{\pi}{2}$), we will need to split the integral such that $I = I_1 - I_2$ where I_1 is integrated between limits η_1 and η_{max} whilst I_2 is integrated between limits η_{max} and η_2 where η_{max} is calculated from $K = \cos\eta \sin\varphi$ with $\varphi = \frac{\pi}{2}$ i.e.:-

` $$\eta_{max} = \cos^{-1}|K| \quad (4.21)$$

The special case of a sphere is obtained by letting $\xi \to \infty$ in which case equation 4.18 becomes:-

$$I(K) = I_0 \int_{\eta_{min}}^{\eta_{max}} \cos\eta \left[\cos\eta \left[1-\left(\frac{K}{\cos\eta}\right)^2\right]^{\frac{1}{2}} \cos\vartheta + \sin\eta \sin\vartheta\right] \left[1 + \frac{\sin^2\eta\left(\frac{K}{\cos\eta}\right)^2}{1-\left(\frac{K}{\cos\eta}\right)^2}\right]^{\frac{1}{2}} d\eta \quad (4.22)$$

4.2 Simulating a Kepler Orbit Disk

Using cylindrical polar co-ordinates, let a point on the surface of the disk have position vector (relative to its centre):-

$$\underline{r} = r\left(\cos\varphi \;\underline{i} + \sin\varphi \,\underline{j}\right) \quad (4.23)$$

Assume that we are viewing the disk in the *ik* plane at elevation angle ϑ to the *ij* plane i.e. from a direction with unit vector:-

$$\underline{\hat{d}} = \cos\vartheta \;\underline{i} + \sin\vartheta \,\underline{k} \quad (4.24)$$

We will also assume that it is rotating anti-clockwise i.e with an angular velocity $\underline{k}$, therefore $\underline{k}$ is the unit normal to the disk.

For circular Kepler orbits we know the velocity varies with the orbit radius according to:-

$$v = \sqrt{\frac{GM}{r}} \tag{4.25}$$

Therefore given the inner radius and the velocity at this radius, v_0 and $r_0 (\equiv 1)$ respectively, we can write:-

$$v = v_0 \sqrt{\frac{1}{r}} \tag{4.26}$$

Therefore

$$\omega(r) = \frac{v}{r} = v_0 \sqrt{\frac{1}{r^3}} \tag{4.27}$$

For a disk of outer radius r_1 the total intensity received is given by:-

$$I = \int_0^{2\pi} \int_1^{r_1} I(r,\varphi)\, dr d\varphi = I_0 \int_0^{2\pi} \int_1^{r_1} \underline{\hat{d}}.\underline{k}\, r dr d\varphi = \pi(r_1^2 - 1) I_0 \sin\vartheta \tag{4.28}$$

Where:-

$$I(r,\varphi) = r I_0 \sin\vartheta \tag{4.29}$$

We need to integrate the function I along a contour of constant line of sight velocity to determine the intensity of light received at a Doppler shift appropriate to that velocity.

The velocity at a given point $\underline{r}$ on the disk is given by:-

$$\underline{v} = \underline{\omega} \wedge \underline{r} = r \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega(r) \\ \cos\varphi & \sin\varphi & 0 \end{vmatrix} = v_0 \sqrt{\frac{1}{r}} \left(-\sin\varphi\, \underline{i} + \cos\varphi\, \underline{j} \right) \tag{4.30}$$

Therefore, equating the line of sight velocity to the Doppler shift we have:-

$$\underline{v}.\underline{\hat{d}} = -v_0 \sin\varphi \cos\vartheta \sqrt{\frac{1}{r}} = \frac{\Delta\lambda c}{\lambda_0} \tag{4.31}$$

or:-

$$K \equiv \frac{-\Delta\lambda c}{v_0 \cos\vartheta \lambda_0} = \sqrt{\frac{1}{r}} \sin\varphi \tag{4.32}$$

Where *K* is a, dimensionless, constant for a given Doppler shift $\Delta\lambda$ and we now need to evaluate the line integral

$$I(K) = \int_{\theta_0}^{\theta_1} I(r,\varphi) \frac{dl}{d\varphi} d\varphi \tag{4.33}$$

Where, using equation (4.32) the line element differentiated wrt *r* is given by:-

$$\frac{dl}{dr} = \sqrt{1 + \left(r\frac{d\varphi}{dr}\right)^2} = \frac{1}{2}\sqrt{\frac{4-3rK^2}{1-rK^2}} \tag{4.34}$$

So finally, integrating over both the front and rear quadrants we have:-

$$I(K) = I_0 \sin\vartheta \left\{ \int_1^{r_{max}} r\sqrt{\frac{4-3rK^2}{1-rK^2}}\,dr + \int_{r_{max}}^{r_{min}} r\sqrt{\frac{4-3rK^2}{1-rK^2}}\,dr \right\} \tag{4.35}$$

The limit $r_{max} = \min(r_1, r_2)$ where r_2 is defined from 4.32 with $\varphi = \frac{\pi}{2}$ and $r_2 \equiv \frac{1}{K^2}$. Whilst the limit r_{min} represents the "shadow" of the star on the rear quadrant of the disk and needs a bit more work to define.

Assuming the star is represented as an oblate spheroid the mathematics of section 4.1 applies and in particular on the visible limb we have $\underline{\hat{n}}.\underline{\hat{d}} = 0$ which results in the relation given in equation (4.19) which we can re-write as:-

$$\tan\eta = -\frac{\cos\varphi \tanh\xi}{\tan\vartheta} = -\frac{r_P}{r_E}\frac{\cos\varphi}{\tan\vartheta} = -\frac{\cos\varphi}{\mathrm{O_b}\tan\vartheta} \tag{4.36}$$

Where we have used equation (4.3) to arrive at the last expression and $\frac{r_E}{r_P}$ is the "oblateness" ratio $\mathrm{O_b}$.

Thus we can determine:-

$$\cos\eta = \frac{1}{\sqrt{1+\left(\frac{\cos\varphi}{\mathrm{O_b}\tan\vartheta}\right)^2}} \quad \text{and} \quad \sin\eta = -\frac{1}{\sqrt{1+\left(\frac{\mathrm{O_b}\tan\vartheta}{\cos\varphi}\right)^2}} \tag{4.37}$$

Now a point on the visible limb satisfies equation (4.36) and has position vector given by equation (4.6) i.e:-

$$\underline{r_l} = \cos\eta\cos\varphi\ \underline{i} + \cos\eta\sin\varphi\,\underline{j} + \tanh\xi\,\sin\eta\,\underline{k} \tag{4.38}$$

So it follows that the projection of $\underline{r_l}$ along direction $\underline{d}$ intersects the disk at the point:-

$$\underline{r_p} = -\left(\frac{\cos\varphi}{\sqrt{1+\left(\frac{\cos\varphi}{\mathrm{O_b}\tan\vartheta}\right)^2}} + \frac{1}{\mathrm{O_b}\tan\vartheta\sqrt{1+\left(\frac{\mathrm{O_b}\tan\vartheta}{\cos\varphi}\right)^2}}\right)\underline{i} + \frac{\sin\varphi}{\sqrt{1+\left(\frac{\cos\varphi}{\mathrm{O_b}\tan\vartheta}\right)^2}}\underline{j} \tag{4.39}$$

Note that, for the back of the disk, $\cos\varphi$ is –ve hence the choice made for the signs of $\cos\eta$ and $\sin\eta$ in (4.37). With this sign choice both terms in the expression for the component along the $\underline{i}$ direction of $\underline{r_p}$ are -ve.

The magnitude $\left|\underline{r_p}\right| \equiv r$ reduces to:-

$$r = \sqrt{1 + \left[\frac{1+(O_b\tan\vartheta)^2}{\cos^2\varphi+(O_b\tan\vartheta)^2}\right]\left(\frac{\cos\varphi}{O_b\tan\vartheta}\right)^2} \tag{4.40}$$

Note: if we set $O_b = 1, \varphi = \pi$ and $\vartheta = \frac{\pi}{4}$ then equation 4.40 yields $r = \sqrt{2}$ as required of a 45 degree tangent piercing the equatorial plain of a unit sphere.

Now using equation (4.32) we can eliminate φ in (4.40) to yield:-

$$K^2r^3 - [1 + (O_b \tan\vartheta)^2]r^2 - \left[\frac{1+2(O_b \tan\vartheta)^2}{(O_b \tan\vartheta)^2}\right]K^2r + \left[\frac{\{1+(O_b \tan\vartheta)^2\}^2}{(O_b \tan\vartheta)^2}\right] = 0 \qquad (4.41)$$

Thus given a value for K we can determine r_{min} by solving equation 4.41 and choosing the root that lies between $r = 1$ and $r_{K=0}$ where, setting $K = 0$ in equation (4.41):-

$$r_{K=0} = \sqrt{\frac{[1+(O_b \tan\vartheta)^2]}{(O_b \tan\vartheta)^2}} \qquad (4.42)$$

Note: if we set $O_b = 1$ and $\vartheta = \frac{\pi}{4}$ then equation (4.42) again yields $r = \sqrt{2}$ as we would expect.

4.2.3 Non-Uniformly Emitting Kepler Orbit Disk

It is possible to include a dimensionless function of r into (4.35) to simulate a disk which varies in emission intensity radially over its surface with peak intensity at radius r_p:-

$$I(K) = I_0 \sin\alpha \left\{ \int_1^{r_{max}} f(r) r \sqrt{\frac{4-3rK^2}{1-rK^2}}\, dr + \int_{r_{max}}^{r_{min}} f(r) r \sqrt{\frac{4-3rK^2}{1-rK^2}}\, dr \right\} \qquad (4.43)$$

The function $f(r) = \left(\frac{r_p}{r}\right)^{-n}$ if $r_p > r$ and $f(r) = \left(\frac{r}{r_p}\right)^{-n}$ if $r_p \le r$ has been implemented in the custom software.

5.0 Convolution of Two Distributions

Given a histogram starting distribution vector (V_0) with known (not necessarily uniform) bin widths $(\Delta\lambda_i)$ we can apply a second spreading distribution to yield the resultant distribution vector (V_1) via the matrix operation:-

$$MV_0 = V_1 \qquad (5.1)$$

where $m_{ij} = D(\lambda_j - \lambda_i)\left(\frac{\Delta\lambda_j}{\Delta\lambda_i}\right)$ and D is the second distribution function.

6.0 Saturation Effects in Absorption Lines

In this section we will first justify and describe the linear absorption model that we shall use assuming a single layer photosphere in thermal equilibrium. Next we will relate the absorption line profile to the dynamics of the absorbing atoms in the photosphere and finally obtain a relationship between the amount of absorption occurring between different lines of a series.

6.1 The Absorption Model

The principles behind this model can best be understood if we imagine isolating a section of a stellar photosphere in an insulating box with perfectly reflecting walls - as far as the photosphere's Planckian photon field is concerned. The walls are however perfectly transparent to all photons from an external Planckian source of the same temperature. If the external source is viewed through the box then we assume only those photons that suffer no absorption emerge from the front face of the box. Any absorbed photons from the external source are scattered and emerge from other faces of the box. Thus from the side of the box we would see an emission spectrum whilst the front face would present an absorption spectrum.

This configuration may seem somewhat contrived but such is the power of assuming thermal equilibrium that, as the configuration could occur and everything "adds up", then it must be indistinguishable from other possible configurations. The downside is of course that in reality not all, and possibly few, photospheres will be well modelled by a single layer in thermal equilibrium. However by comparing real spectra to this simple model it should be possible to speculate on the reasons for any deviation. More accurate multi-layer models exist but those will be left to the professionals.

6.2 Linear Absorption Model

The *i* to *j* principle quantum level transition absorption line profile (*j*>*i*) at a given temperature *T*, expressed as a photon number flux per unit wavelength, will be represented by the function $P_{ij}(\lambda, x)$. The change in $P_{ij}(\lambda, x)$ when passing through a unit area slab of thickness *dx* at position *x* is given by:-

$$dP_{ij}(\lambda, x) = -\sigma_{ij} P_{ij}(\lambda, x) N_i(\lambda) d\lambda dx \qquad (6.1)$$

P_{ij} is a function of wavelength λ by virtue of the dynamics of the stellar photosphere (pressure, rotation and thermal motion). This dynamics is represented by the function $N_i(\lambda)$ which is the number of absorbing atoms per cubic metre per unit wavelength in the *i*th principle quantum state and able to transition to the *j*th state by absorbing a photon of wavelength λ. The final factor σ_{ij} is a "capture cross-section" and represents the probability of absorbing a photon to transition from the *i*th to *j*th state and is defined in the rest-frame of an atom where we always have $\lambda = \lambda_{ij}$.

Equation (6.1) can be integrated (w.r.t. x) to yield:-

$$\hat{P}_{ij}(\lambda, t) \equiv \frac{P_{ij}(\lambda, t)}{P_{ij}(\lambda, 0)} = e^{-\sigma_{ij} t N_i(\lambda) d\lambda} \qquad (6.2)$$

Where *t* is the thickness of the photosphere and we have normalised the photon number to a continuum of 1.0. The photon number at *x = 0* is given by the Planck function in the form of a number flux (see section 6.4) i.e:-

$$P_{ij}(\lambda, 0) = \mu(\lambda, T) d\lambda = \frac{2\pi c}{\lambda^4} \frac{d\lambda}{e^{\frac{hc}{kT\lambda}} - 1} \quad \mathrm{m^{-2}\,s^{-1}} \qquad (6.3)$$

$\hat{P}_{ij}(\lambda, t)$ in fact represents the measured normalised absorption profile, in the remainder of this section we will not indicate the photosphere thickness explicitly and just refer to the normalised photon *i* to *j* absorption profile as $P_{ij}(\lambda)$.

Note that:-

$$\int N_{ij}(\lambda) d\lambda = N_i \qquad (6.4)$$

Where N_i is the total number of atoms m^{-3} in state *i*. Now defining a scale factor s_j using:-

$$s_j \int \hat{N}_{ij}(\lambda) d\lambda = N_i \tag{6.5}$$

Where $\hat{N}_{ij}(\lambda_{ij}) = 1$, we can write (6.2) as:-

$$P_{ij}(\lambda) = e^{-\sigma_{ij} t s_j \hat{N}_{ij}(\lambda) d\lambda} \tag{6.6}$$

Next we shall modify the notation further by defining an "equivalent emission line" via:-

$$E_{ji}(\lambda) \equiv \hat{N}_{ij}(\lambda) \tag{6.7}$$

$E_{ji}(\lambda)$ is the normalised emission line profile that would be seen if we could selectively observe the *j* to *i* emission process within the star's photosphere.

So we can write:-

$$P_{ij}(\lambda) = e^{-\sigma_{ij} t s_j E_{ji}(\lambda) d\lambda} \tag{6.8}$$

And as $E_{ji}(\lambda_{ij}) = 1$ it follows that:-

$$P_{ij}(\lambda_{ij}) = e^{-\sigma_{ij} s_j t d\lambda} \tag{6.9}$$

Taking natural logarithms of (6.8) and (6.9) we can deduce:-

$$E_{ji}(\lambda) = \frac{Ln\left(P_{ij}(\lambda)\right)}{Ln\left(P_{ij}(\lambda_{ij})\right)} \tag{6.10}$$

and therefore:-

$$P_{ij}(\lambda) = P_{ij}(\lambda_{ij})^{E_{ji}(\lambda)} \tag{6.11}$$

We can use (6.10) to generate an equivalent emission line corresponding to a particular measured absorption line. This emission line can then be analysed to produce a model of the photosphere dynamics (Temperature, Pressure and Rotation). The resulting model can then be used to generate the equivalent emission line for a second line in the spectral series. To complete the process (6.11) can be used to predict the expected absorption line. The following sub-sections will fill in the details of this analysis method.

6.3 Relation between two lines of a series

For a second line of a spectral series we can write (6.8) as:-

$$P_{ik}(\lambda) = e^{-\sigma_{ik} s_k t E_{ki}(\lambda) d\lambda} \tag{6.12}$$

Taking the natural logarithm of (6.12) and it's counterpart for level *k* we can deduce:-

$$P_{ik}(\lambda_k) = \left[P_{ij}(\lambda_j)\right]^{\frac{\sigma_{ik} s_k E_{ki}(\lambda_k) d\lambda_k}{\sigma_{ij} s_j E_{ji}(\lambda_j) d\lambda_j}} \tag{6.13}$$

Where we have given the wavelength symbol a single subscript to indicate the different wavelength variables. From (6.5) we deduce that for any two lines of a spectral series

$$s_j \int E_j(\lambda)d\lambda = s_k \int E_k(\lambda)d\lambda = N_i \qquad (6.14)$$

or

$$s_j w_j = s_k w_k = N_i \qquad (6.15)$$

Where w_j is the equivalent width of the *j* emission line which is equal to the area of the normalised line as obtained by integrating with respect to wavelength. Substituting into (6.13) we finally obtain:-

$$P_{ik}(\lambda_{ik}) = \left[P_{ij}(\lambda_{ij})\right]^{\frac{\sigma_{ik} w_k d\lambda_k}{\sigma_{ij} w_j d\lambda_j}} \qquad (6.16)$$

as $E_{ki}(\lambda_{ik}) = 1$ by definition. All factors on the right-hand side of equation (6.16) are now known except for the capture cross-sections which we will determine in the following sub-section.

Note that from (6.9) we have:-

$$s_j t = \frac{-Ln[P_{ij}(\lambda_{ij})]}{\sigma_{ij} d\lambda} \qquad (6.17)$$

Substitution from (6.15) allows us to determine that:-

$$N_i t = \frac{-w_j Ln[P_{ij}(\lambda_{ij})]}{\sigma_{ij} d\lambda} \qquad (6.18)$$

So once σ_{ij} is determined we can also obtain a value for the number of atoms m^{-3} in state *i* multiplied by the photosphere thickness i.e. the column density.

6.4 Einstein Coefficients

Capture and emission processes between two atomic levels with principle quantum numbers i and j (j > i) are governed by the Einstein coefficients. Einstein coefficients can be calculated in various sets of variables we will use:-

- N_i units m^{-3}, is the number density of hydrogen atoms with an electron in the ith energy level at a given point in a photosphere.
- g_i is the electron degeneracy of the *i*th energy level.
- P_{ij} units m^{-2} s^{-1} is the number flux of photons that can induce the i to j transition.
- $\mu(\lambda, T)$ units m^{-2} s^{-1}, is the Planck distribution photon number flux at temperature T and transition wavelength λ.
- A_{ji} with units s^{-1}, is the Einstein coefficient for spontaneous photon emission from the electron n=j to n=i level transition (j>i).
- B_{ji} units m^{2}, is the Einstein coefficient for electron stimulated emission from the n=j to n=i level.
- B_{ij} units m^{2}, is the Einstein coefficient for photon capture resulting in an electron n=i to n=j transition.

The A and B Einstein coefficients are fundamental properties of their associated atom and whilst the A coefficients can be measured or calculated using quantum mechanics, the B coefficients are normally derived from the A's by considering how atoms in thermal equilibrium interact with a thermal equilibrium radiation field of the same temperature. Under these conditions the electron population of the atomic levels are known allowing the B coefficients to be calculated. To obtain the relation between the A and B Einstein coefficients note that the rates of change of level populations in an atom can be expressed as:-

$$-\frac{dN_j}{dt} = \frac{dN_i}{dt} = A_{ji}N_j - B_{ij}P_{ij}N_i + B_{ji}P_{ij}N_j \qquad (6.19)$$

In equilibrium $-\frac{dN_j}{dt} = \frac{dN_i}{dt} = 0$ therefore we can deduce:-

$$\frac{N_j}{N_i} = \frac{B_{ij}P_{ij}}{A_{ji}+B_{ji}P_{ij}} \qquad (6.20)$$

In thermal equilibrium detailed balance requires:-

$$g_i B_{ij} = g_j B_{ji} \qquad (6.21)$$

which together with the Boltzmann relation:-

$$Z(\lambda, T) \equiv \frac{N_j}{N_i} = \frac{g_j}{g_i} e^{\frac{-hc}{kT\lambda_{ij}}} \qquad (6.22)$$

allows us to deduce in units of m^2:-

$$g_i B_{ij} = \frac{g_j A_{ji}}{P_{ij}\left(e^{\frac{hc}{kT\lambda_{ij}}}-1\right)} \qquad (6.23)$$

To proceed further we need an expression for P_{ij}, now the Planck function can be expressed in two forms:-

1. Energy density $\rho(\lambda, T)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{\frac{hc}{kT\lambda}}-1}$ J m^{-3}
2. Energy flux $\eta(\lambda, T)d\lambda = \frac{2\pi hc^2}{\lambda^5}\frac{d\lambda}{e^{\frac{hc}{kT\lambda}}-1}$ Wm^{-2}

It seems most appropriate in our case to use form 2 as our absorption model is framed in terms of a flow of photons through a photosphere. Dividing the Energy flux by the photon energy $\frac{hc}{\lambda}$ yields the photon number flux:-

$$\mu(\lambda, T)d\lambda = \frac{2\pi c}{\lambda^4}\frac{d\lambda}{e^{\frac{hc}{kT\lambda}}-1} \text{ m}^{-2}\text{ s}^{-1} \qquad (6.24)$$

Multiplying by a Dirac delta probability function and integrating over all wavelengths yields the result:-

$$P_{ij} = \mu(\lambda_{ij}, T) \qquad \text{m}^{-2}\text{ s}^{-1} \qquad (6.25)$$

Substituting from (6.24) into (6.23) and using (6.25) we obtain:-

$$g_i B_{ij} = \frac{g_j A_{ji}}{\mu(\lambda_{ij},T)\left(e^{\frac{hc}{kT\lambda_{ij}}}-1\right)} = g_j \frac{A_{ji}\lambda_{ij}^4}{2\pi c} \qquad \mathrm{m}^2 \tag{6.26}$$

We can now relate the Einstein *B* coefficient for an *i* to *j* capture event to the corresponding *A* spontaneous emission constant:-

$$B_{ij} = \frac{g_j}{g_i}\frac{A_{ji}\lambda_{ij}^4}{2\pi c} \qquad \mathrm{m}^2 \tag{6.27}$$

The *A* Einstein coefficients are readily available in the literature from detailed quantum calculations, Table 6.1 lists them for transitions of the Hydrogen Balmer series.

Table 6.1 Hydrogen Einstein A_{ji} Coefficients 10^8 s^{-1}

i\j	2	3	4	5	6
1	4.69669	0.55727384	0.1277960	0.0412330	0.0164334
2	0	0.44082910	0.0841572	0.0252935	0.0097278
3	0	0	0.0898228	0.0219982	0.0077796
4	0	0	0	0.0269813	0.0077078
5	0	0	0	0	0.0102497

6.5 Relationship between the Einstein Coefficients and σ_{ij}

Although the B_{ij} have units of area they are not the capture cross-sections we seek and indeed if we substitute their values into (6.16) predictions of absorption are in error by many orders of magnitude. In addition the relative absorption amplitudes are observed experimentally to be temperature dependent which the $B_{ij}$are most definitely not. In this section we will derive the relationship between the Einstein Coefficients and σ_{ij}.

To proceed note that we must have:-

$$\frac{\sigma_{ik}\mu(\lambda_{ik},T)}{\sigma_{ij}\mu(\lambda_{ij},T)} = \frac{N_k}{N_j} \tag{6.28}$$

if (6.28) did not hold the level populations over time would depart from their equilibrium values. Thus:-

$$\frac{\sigma_{ik}}{\sigma_{ij}} = \frac{\mu(\lambda_{ij},T)N_k}{\mu(\lambda_{ik},T)\,N_j} \tag{6.29}$$

Note both the σ_{ij} and the B_{ij} are functions of the level population and photon field variables and can be explicitly related if desired.

From (6.29) we can deduce:-

$$\sigma_{ij} = \frac{K_i}{\mu(\lambda_{ij},T)}\frac{N_j}{N_i} \tag{6.30}$$

Where K_i, for all lines of a given spectral series, is a constant with units s^{-1}. We will define the K_i in terms of the Einstein coefficients via:-

$$\mathrm{K_i} \equiv \frac{\alpha}{N_i} \sum_{k=i+1}^{\infty} A_{ki} N_k \tag{6.31}$$

where α is the fine structure constant and we have ignored the effects of stimulated emission. So we can finally write:-

$$\sigma_{ij} = \frac{\alpha}{N_i^2} \frac{N_j}{\mu(\lambda_{ij}, T)} \sum_{k=i+1}^{\infty} A_{ki} N_k \qquad \mathrm{m}^2 \tag{6.32}$$

Equation (6.32) together with (6.22) and (6.24) allow all capture cross-sections to be calculated for any given temperature. In practice the summation in equation (6.32) decreases rapidly with index *k* and is therefore convergent, it is truncated at *k = 20* within the software implementation.

Whilst (6.30) has been fully justified (6.31) does need more consideration. The summation term in (6.31) represents the total emission rate and so is a reasonable factor to employ as a "Lego brick" to construct the factor $\mathrm{K_i}$. Including this factor means the capture cross-sections are being expressed as proportions of the total emission rate with those proportions being determined by the appropriate level population and the Plankian photon flux.

Regarding the inclusion of the factor α, this factor often appears in equations describing the interaction between photons and electrons e.g the "Oscillator Strength" , so is again a reasonable factor to include. Up to this point these observations are the only justifications for choosing to define $\mathrm{K_i}$ as written in (6.31). However, we will demonstrate in the next section that the capture cross-sections so defined lead to acceptable predictions for known properties of the Sun.

Note, if we wish to include the effect of stimulated emission then (6.32) would become:-

$$\sigma_{ij} = \frac{\alpha}{N_i^2} \frac{N_j}{\mu(\lambda_{ij}, T)} \sum_{k=i+1}^{\infty} A_{ki} \left(1 + \frac{\mu(\lambda_{ik}, T)\lambda_{ik}^4}{2\pi c}\right) N_k \qquad \mathrm{m}^2 \tag{6.33}$$

7.0 Comparing Theory with Known properties of the Sun

First a little Thermodynamics, the perfect gas law states that:-

$$PV = nRT \tag{7.1}$$

where *P* is pressure, *V* is volume, *T* is absolute temperature, n is the number of moles of the particles, *R* (= 8.31441) is the molar gas constant therefore:-

$$P = \frac{n}{V} RT \equiv n_v RT \tag{7.2}$$

where n_v is the number of moles of the particles per unit volume, defining *N* as the number of particles per unit volume we have:-

$$P = \frac{N}{N_A} RT \tag{7.3}$$

where N_A is Avogadro's number (= 6.022045e23). An alternative way of writing the same equation is:-

$$P = NkT \tag{7.4}$$

Where k is Boltzmann's constant (=1.380662e-23).

A given stellar line profile in the Hydrogen Balmer series can be modelled using the theory of sections 2, 3 and 4 thus obtaining values for the temperature T , from the Planckian continuum, and pressure from the Lorentz distribution half width $\frac{\Gamma'}{2}$ via equations (3.9) and (7.4) given a value for the impact parameter ρ.

We can then use Saha's equation to determine the number of neutral atoms N_I and ionised atoms N_{II} . Assuming the number of free electrons (n_e) equals the number of ionised Hydrogen atoms i.e. $n_e = N_{II}$, Saha's equation states:-

$$N_{II}^2 = \frac{N_I}{\Lambda^3} \exp\left(-\frac{E_{ion}}{kT}\right) \tag{7.5}$$

where E_{ion} is the ionisation energy of, in this case, Hydrogen (13.6eV), Λ is the electron thermal de Broglie wavelength $\left(\Lambda = \sqrt{\frac{h^2}{2\pi m_e kT}}\right)$ and m_e is the electron rest mass. Note that $N_I = N - N_{II}$ therefore (7.5) can be solved as a quadratic in N_{II}.

Solar Photosphere as a Function of Depth

Depth (km)	% Light from this Depth	Temperature (K)	Pressure (bars)
0	99.5	4465	6.8×10^{-3}
100	97	4780	1.7×10^{-2}
200	89	5180	3.9×10^{-2}
250	80	5455	5.8×10^{-2}
300	64	5840	8.3×10^{-2}
350	37	6420	1.2×10^{-1}
375	18	6910	1.4×10^{-1}
400	4	7610	1.6×10^{-1}

Source: Fraknoi, Morrison, and Wolf, *Voyages through the Universe*

Table 7.1: Published data on the Solar photosphere

Only the neutral hydrogen atoms produce spectral lines and of these only those in principle quantum state *i=2* are the base level for the Balmer series, Boltzmann's equation states:-

$$N_{i=2} = \frac{N_I}{4} \exp\left[\frac{-hc}{kT\lambda_{12}}\right] \tag{7.6}$$

where λ_{12} is the Lyman α wavelength 1216 A. Finally having obtained a value for $N_{i=2}$ as a function of the impact parameter ρ we can use equation (6.19), with *i* = 2, to obtain a corresponding value for the thickness of the photosphere as a function of ρ. In a separate document I detail the analysis of the solar Hydrogen Balmer alpha and beta lines. With an impact parameter ρ = 4.0e-10, the solar photosphere was calculated to have a thickness of 400.41 km (387.39 km when stimulated emission is included, see equations (6.32) and (6.33)) and a pressure of 0.1135 Bar. This result compares remarkably well with published data given in table (7.1). Note also that the value of the column density depends on the absolute value of the capture cross-section via equation (6.18) so the good agreement lends strong support to the definition in equation (6.33).

7.1 Element Relative Abundances

Elements other than Helium can be assumed to be present in low concentrations relative to Hydrogen therefore we can set $n_e = N_{II}^H$ where N_{II}^H is the number of ionised Hydrogen atoms. In this case Saha's equation becomes:-

$$N_{II} = \frac{N}{N_{II}^H \Lambda^3 \exp\left(\frac{E_{ion}}{kT}\right)+1} \qquad (7.7)$$

Where again $N_I = N - N_{II}$, N being the concentration of the element in the chosen series base state.

To obtain an abundance estimate we can use equation (6.18) to relate N to properties of a measured series line, the photon capture cross-sections and the width of a photosphere as obtained by modelling a Hydrogen Balmer line i.e.:-

$$N_I = \frac{-w_j Ln[P_{ij}(\lambda_{ij})]}{t\sigma_{ij} d\lambda} \qquad (7.8)$$

With capture cross sections calculated from equation (6.32) assuming the relevant Einstein A coefficients are known. For any given element its actual concentration would be related to the series base state concentration N via the Boltzmann equation (6.22).

7.2 Estimating Photosphere Pressure from Surface Gravity

The pressure at the base of a photosphere must support the column of matter above it, therefore we can write:-

$$P = g_s \rho_c \qquad \text{N m}^{-2} \qquad (7.9)$$

Where P is the pressure, g_s is the surface gravity (m s^{-2}) and ρ_c is the column mass density (kg m^{-2}). For the Sun $g_s = 273.7$ m s^{-2}.

Equations (6.18) and (6.33) enable us to calculate a value for the compound property $N_2 t$ i.e. the column number density of atoms in the *i =2* principle quantum state and therefore using the Boltzmann relation $\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{\frac{-hc}{kT\lambda_{12}}}$ we can write for the compound property $N_I t$:-

$$N_I t \approx \frac{N_2 t}{4} e^{\frac{hc}{kT\lambda_{12}}} \qquad (7.10)$$

Where N_I is the number density of neutral atoms in the photosphere. We now need to use Saha's equation (7.5) to determine the ionised atom number density N_{II} but as equation (7.5) is nonlinear we have to make this calculation as a function of the photosphere thickness t given that $N_I = (N_I t)/t$. Therefore:-

$$N_{II}(t) = \sqrt{\frac{N_I(t)}{\Lambda^3} \exp\left(-\frac{E_{ion}}{kT}\right)} \qquad (7.11)$$

We can now write:-

$$\rho_c(t) = M\{N_I + N_{II}(t)\}t \tag{7.12}$$

where *M* is the mass of the identical impacting atoms.